TOPIC PROPOSAL
TOPICS IN 3-MANIFOLDS

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Introduction

A central problem in geometry and topology is to understand the submanifolds of a given manifold M; in particular, those of codimension 1. When M is a 3-manifold, the situation is particularly nice: the codimension-1 submanifolds are surfaces, which have their own rich and beautiful theory, and M has a geometric structure, a canonical decomposition into pieces which each carry one of Thurston's eight model geometries. Moreover, these two aspects of M are deeply related.

A basic problem is to take an *a priori* non-embedded surface (e.g., a disk realizing a nullhomotopic loop) and produce an embedded surface with similar properties. Indeed, the Sphere Theorem (Theorem 1.14) and the Loop Theorem (Theorem 1.7) provide instances of this principle. This illustrates a wider theme in 3-manifolds - weak algebraic assumptions lead to surprising geometric and topological rigidity. In the case of the Sphere theorem, Stallings' proof uses the fact that the Cayley graph associated to the fundamental group of the manifold has multiple ends to produce an embedded essential sphere.

The geometric side of the picture tells a similar story. In dimensions greater than two, Mostow rigidity promotes an isomorphism between fundamental groups of complete, finite-volume hyperbolic manifolds to an isometry between the spaces. Thus such a manifold carries a unique hyperbolic structure, and for this class of manifolds, the fundamental group is a complete topological and geometric invariant.

As a tool for understanding the embedded surfaces of a manifold, Thurston introduced a norm on $H_2(M;\mathbb{R})$, roughly measuring the minimal topological complexity of an embedded surface representing a homology class. Thurston's norm provides an important invariant in the class of knot complements, manifolds which are the complement of a knot K embedded in a manifold such a S^3. Here the surfaces of interest are the Seifert surfaces - those with boundary K. The minimal genus of a Seifert surface associated to K is the *genus* of K. The minimal Seifert surface also realizes the Thurston norm of its homology class in $H_2(M;\mathbb{R})$.

Computing the Thurston norm is not an easy task, but here algebra saves the day. Work of Agol and Dunfield in [1] shows the Thurston norm of a large class of knots is realized by a variant on the Alexander polynomial with twisted homology. Moreover, this polynomial is computationally easy to produce. This follows from earlier work by Friedl and Kim in [3] which shows a Seifert surface realizes its class' Thurston norm if the knot complement, cut open along the surface, is a twisted homology product.

Our goal is to explore the topology and geometry of 3-manifolds, highlighting the connections to group theory and the ways group theoretic information provides a natural context for understanding and answering problems in geometry and topology. In this, we include a discussion of Bass-Serre theory, which provides especially useful tools in this context for understanding the relationship between groups and 3-manifolds.

1. 3-manifold topology

We begin in this section with the basics of three-dimensional topology, presenting some of the foundational tools in 3-manifold topology. These results can be found in [2].

1.1. Some examples of 3-manifolds.

Example 1.1. 3-Manifolds can be constructed by gluing faces of tetrahedra in pairs. Such a complex is a 3-manifold if and only if its Euler characteristic is zero. Moreover, by Moise, every 3-manifold can be triangulated and so realized in this way.

Date: December 1, 2015.

Example 1.2. A decomposition of a 3-manifold into two handlebodies identified along the boundaries is known as a *Heegaard splitting*. Every oriented 3-manifold admits such a decomposition, for example, by thickening the 1-skeleton and dual 1-skeleton of a triangulation. The *genus* of a heegaard splitting is the genus of the boundary surfaces of the handlebodies. Waldhausen proved 3-sphere admits a unique Heegaard splitting of every genus.

Example 1.3. Every oriented 3-manifold can be obtained by (Dehn) surgery on a link in the 3-sphere (see Section 1.5 for a definition). For, any two Heegaard splittings of the same genus are related by an element of the mapping class group of the splitting surface, which is generated by Dehn twists; such twists can be realized by surgery on a knot.

Example 1.4 (Lens spaces). Lens spaces are obtained from the 3-sphere (thought of as the unit sphere in $\mathbb{C}^2$) by taking the quotient of a freely acting finite cyclic group. Lens spaces have genus 1 (as a Heegaard splitting). The 3-sphere is a (trivial) example of a lens space.

Example 1.5 (Seifert fibered spaces). A foliated space with S^1 leaves is known as a Seifert fibered spaces. The local structure is a product away from finitely many leaves whose neighborhoods look like the suspension of a rotation of finite order on a disk. Lens spaces are Seifert fibered. A Seifert fibered 3-manifold which is not covered by the 3-sphere, and which has orientable fibers, has a central $\mathbb{Z}$ in its fundamental group. We can think of a Seifert fibered space as a circle bundle over a 2-dimensional *orbifold*. When this orbifold is S^2 with at most three critical (or *orbifold*) points, the Seifert space is *small*.

Example 1.6 (Surface bundles). If S is a surface and $\varphi : S \to S$ is a homeomorphism, we can form the associated surface bundle $M_\varphi := S \times [0,1]/(s,1) \sim (\varphi(s),0)$. This depends only on the class of φ in the mapping class group of S. We say M_φ *fibers over the circle* and the map φ is the *monodromy* of the bundle.

1.2. **The Loop Theorem and Dehn's Lemma.** One of first tools in 3-manifold topology is the Loop Theorem, which relates relatively weak assumptions on the fundamental group to the existence of an embedded disk.

Theorem 1.7 (Loop Theorem). *Let M be a 3-manifold, and suppose $\pi_1(\partial M) \to \pi_1(M)$ induced by inclusion has a nontrivial kernel. Then there is a properly embedded disk D in M such that ∂D is essential in $\pi_1(\partial M)$.*

Corollary 1.8 (Dehn's Lemma). *A proper map of a disk D into a 3-manifold M which is an embedding on a collar neighborhood of the boundary can be replaced by a proper embedding which agrees on a neighborhood of the boundary.*

The Loop Theorem was proved by Papakyriakopolous using a *tower argument*. One considers a regular neighborhood of the image of a disk, and passes to a nontrivial double cover (which exists for homological reasons) and a lift of the disk to the cover which has simpler self-intersections. At the top of the tower one obtains an embedded disk. Then one inductively pushes the disk down the tower, applying cut-and-paste at each stage to get an embedded disk. Since the covering maps have degree 2, the images at each stage have only double arcs or curves of intersection.

1.3. **Incompressible surfaces and Kneser's Lemma.**

Definition 1.9. A 2-sided surface S in M is *incompressible* if no essential simple closed curve on S bounds an embedded disk in $M - S$.

We use to Loop Theorem to translate incompressiblity into the following desirable topological property.

Theorem 1.10 (Kneser's Lemma). *An incompressible surface is π_1-injective.*

Proof. If S is not π_1-injective, some loop γ is essential in S, but trivial in $\pi_1(M)$, so γ bounds a disk $D \subseteq M$. Using the fact that S is two-sided and embedded in M, γ can be pushed off to one side of S, and D then perturbed to intersect S in disjoint simple closed curves. We can eliminate intersections of D and S by working from the inside out: if an innermost intersection curve is inessential in S, its interior in D can be swapped for the disk it bounds in S, then pushed off S to remove this intersection. Repeating, we arrive either at a curve of intersection which is essential in S, or we eliminate all intersections between S and D. In either case, we have produced a curve which is essential in S but bounds a disk in $M - S$. Then apply the Loop Theorem to M cut open along S to produce an embedded disk with boundary essential in S. □

Example 1.11. If S is not 2-sided, yet otherwise satisfies the definition of incompressibility, S may not be π_1-injective. A lens space $L(2p, 1)$, $p > 1$, contains a loop which bounds an embedded nonorientable surface. When fully compressed this surface still has a projective plane component. As $\pi_1(L(2p, 1)) = \mathbb{Z}/2p\mathbb{Z}$, this is only π_1-injective if S is a projective plane, but in that case we arrive at a decomposition $\pi_1(L(2p, 1)) = G * \mathbb{Z}/2\mathbb{Z}$, which is impossible for $p > 1$.

Despite this, every 2-sided surface is homologous to an incompressible surface.

Example 1.12. Every class $\alpha \in H_2(M; \mathbb{Z})$ can be represented by an incompressible surface. Given a representative 2-sided surface S which is not incompressible, S must contain an essential loop γ which bounds an embedded disk D in $M - S$. S can then be compressed by cutting along ∂D and gluing in two parallel copies of D, resulting in a surface of lower complexity in the same homology class. If α is nontrivial, this surface is nonempty.

Example 1.13. If $\pi_1(M)$ acts on a tree (in a suitable way) then there is a surface "dual" to an edge which is incompressible.

1.4. **Sphere Theorem.** A 3-manifold is *irreducible* if every embedded 2-sphere bounds a ball, and is *reducible* otherwise. Being reducible is equivalent to being a nontrivial connect sum (note that a reducing sphere is *not* necessarily separating).

Theorem 1.14 (Sphere theorem)**.** *Let M be a closed oriented 3-manifold with $\pi_2(M)$ nontrivial. Then M is reducible.*

Originally proved by Papakyriakopolous using a tower argument similar to his proof of the Loop Theorem, Stallings later gave a simpler proof employing his theory of ends, discussed in Section 3.4.

Proof. (Sketch) Via Poincaré duality, nontriviality of $\pi_2(M)$ implies the existence of at least 2 ends in $\pi_1(M)$. By Theorem 3.14 and Proposition 3.2, this corresponds to a nice action of $\pi_1(M)$ on a tree with finite edge stabilizers. As in Example 1.13, this action gives the existence of an incompressible surface S in M, which by Kneser's Lemma is π_1-injective, not just into $\pi_1(M)$, but into the corresponding edge stabilizer, forcing S to be an essential sphere. $\square$

The Sphere Theorem gives a method of decomposing a 3-manifold into a connect sum of irreducible components. By Kneser finiteness, this decomposition is both finite and unique, as any 3-manifold contains a *finite* collection of disjoint closed incompressible surfaces which divide the manifold into irreducible pieces, and any remaining incompressible surfaces are contained in a product piece.

1.5. **Dehn Surgery.** As stated in Example 1.3, any manifold oriented 3-manifold can be constructed by modifying the 3-sphere in an appropriate way. This modification is *Dehn surgery*, in which an embedded solid torus in M is cut out, then reglued into M by some homeomorphism of the boundary torus. This homeomorphism is determined (up to isotopy) by the image of the meridian of the torus, another essential simple closed curve in T^2. Since $\pi_1(T^2) = \mathbb{Z}^2 = \langle \ell, m \rangle$, the essential closed curves are the curves $p\ell + qm$, where $(p, q) = 1$.

2. 3-manifold geometry

We now turn to the geometry of 3-manifolds, following [6].

2.1. **Geometric structures in dimension 3.** Thurston gives a complete classification of geometric structures in three dimensions. These so-called *model geometries* are defined as a pair (G, X), where X is a simply connected manifold and G a Lie group of diffeomorphisms of X, with the additional condition that G acts transitively on X with compact point stabilizers and is the maximal such group which does so. We also require the existence of a closed manifold with the geometry of (G, X).

There are 8 model geometries in dimension 3. These are classified by the dimension of the point stabilizer:

(1) the isotropic geometries, with 3-dimensional stabilizers — these are the constant-curvature geometries $\mathbb{H}^3$, $\mathbb{S}^3$, $\mathbb{E}^3$;
(2) the fibered geometries, with 1-dimensional stabilizers, $\mathbb{H}^2 \times \mathbb{R}$, $\mathbb{S}^2 \times \mathbb{R}$, Nil, $\widetilde{\mathrm{SL}(2, \mathbb{R})}$; and
(3) the "Anosov" geometry Sol, which has 0-dimensional stabilizers.

In contrast, there are only three model geometries in dimension 2: $\mathbb{H}^2$, $\mathbb{S}^2$, and $\mathbb{E}^2$. Here, since G acts transitively on X, the space must have constant Gaussian curvature, and we are restricted to these three examples.

Proposition 2.1. *All the geometries but $\mathbb{H}^3$ and Sol are Seifert fibered:*

(1) *A circle bundle over a spherical orbifold has $\mathbb{S}^3$ geometry if it has nonzero Euler class, and $\mathbb{S}^2 \times \mathbb{R}$ geometry otherwise;*
(2) *A circle bundle over a Euclidean orbifold has Nil geometry if it has nonzero Euler class, and $\mathbb{E}^3$ geometry otherwise;*
(3) *A circle bundle over a hyperbolic orbifold has $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ geometry if it has nonzero Euler class, and $\mathbb{H}^2 \times \mathbb{R}$ geometry otherwise.*

Surface bundles over the circle have geometry related to algebra, classified by monodromy. The monodromy, if of infinite order, is *reducible* if it fixes the isotopy class of some simple closed curve and *(pseudo)-Anosov* otherwise.

Example 2.2.

(1) Sphere bundles all have $\mathbb{S}^2 \times \mathbb{R}$ geometry;
(2) Torus bundles with finite order monodromy have $\mathbb{E}^3$ geometry, those with reducible monodromy have Nil geometry, those with Anosov monodromy have Sol geometry;
(3) Higher genus surface bundles with finite order monodromy have $\mathbb{H}^2 \times \mathbb{R}$ geometry, those with pseudo-Anosov monodromy have $\mathbb{H}^3$ geometry, those with reducible monodromy contain an essential torus and have a nontrivial JSJ decomposition (see Theorem 2.3).

A natural question is whether any 3-manifold can be given one of these geometric structures, or broken into pieces which carry a geometric structure. Thurston conjectured that a manifold has a canonical decomposition into pieces which each carry one of the eight natural geometric structures; this was later proved by Perelman.

Theorem 2.3 (Geometrization Theorem)**.** *Let M be a closed, connected, oriented 3-manifold.*

(1) *M has a unique prime decomposition*
$$M = M_1 \# M_2 \# \cdots \# M_n \#_k S^1 \times S^2,$$
where each M_i is irreducible.
(2) *Each irreducible component M_i has a JSJ decomposition, a unique minimal way of cutting M_i open along incompressible tori into pieces which are either atoroidal or Seifert fibered.*
(3) *Each component in the decomposition, atoroidal, Seifert fibered, or $S^1 \times S^2$, carries a geometric structure.*

In particular, the atoroidal pieces are all hyperbolic or small Seifert spaces. As illustrated in Proposition 2.1, the geometry of the Seifert pieces is well-understood and highly constrained. This leaves the hyperbolic pieces, which have a rich theory unto themselves.

2.2. **Hyperbolic geometry.** In this section we restrict attention to the hyperbolic setting, beginning with hyperbolic space itself. There are four commonly encountered models for $\mathbb{H}^n$.

The hyperboloid model. The space itself is the positive sheet of the hyperboloid in $\mathbb{R}^{n+1}$ determined by
$$q(x) = x_1^2 + \ldots + x_n^2 - x_{n+1}^2 = -1.$$
Geodesics are the intersection curves of hyperplanes through the origin with the hyperboloid. The isometry group is easiest to see in this setting. The isometries are exactly the linear maps which preserve the bilinear form
$$B(u, v) = u_1 v_1 + \ldots + u_n v_n - u_{n+1} v_{n+1}$$
associated to q. These maps correspond to the quotient $O(n, 1)/(O(n) \times O(1))$.

The Klein projective model. In this case the space is the open unit ball in $\mathbb{R}^n$, with straight lines as geodesics. This model can be obtained from the hyperboloid model by taking the unit n-ball in $\mathbb{R}^{n+1}$

centered at $(0, \dots, 0, -1)$ in the hyperplane $\{x_{n+1} = -1\}$ and projecting points from the hyperboloid onto the ball via the ray through the point from the origin. In this way, all of the geometry of this model is inherited from the hyperboloid model.

The Poincaré unit ball model. The space is the (open) unit ball in $\mathbb{R}^n$. Geodesics are arcs of circles and lines which meet the boundary of the ball orthogonally. In dimension 3, the isometry group is $\mathrm{PSL}(2, \mathbb{C})$, which acts naturally on the boundary S^2 and extends to an action by isometries in the interior.

The Poincaré upper half-space model. This model is closely related to the last. The space is the upper half-space $\mathcal{H}^+ \subset \mathbb{R}^n$ and geodesics are arcs of circles orthogonal to the boundary and vertical lines. This model is obtained from the unit ball model by the Möbius transformation of $\mathbb{R}^n$ taking the boundary sphere to $\partial\mathcal{H}^+ \cup \{\infty\}$.

The isometries of hyperbolic space $\mathbb{H}^n$ are divided into three types of transformations.

Definition 2.4. A hyperbolic isometry g is
(1) *hyperbolic* if g acts as a non-trivial translation along some geodesic;
(2) *elliptic* if g fixes (pointwise) a geodesic; or
(3) *parabolic* if g preserves no geodesic.

The isometries extend continuously to maps on the boundary sphere at infinity. Since an isometry is completely determined by its action on this sphere, this gives another way to understand the isometry group. For simplicity, we work in the Poincaré ball model, so the sphere at infinity is the literal unit $(n-1)$-sphere.

Consider the number of fixed points of an isometry g in the boundary. If g is hyperbolic or elliptic, it must fix two points, the endpoints of the g-invariant geodesic. For g a hyperbolic isometry, this geodesic is unique, for g elliptic, it might not be. An elliptic isometry which fixes multiple geodesics must also fix the unique (hyper)sphere containing them, and so fix infinitely many boundary points. This leaves g parabolic. By Brouwer's fixed point theorem, g must fix some point, which necessarily lies on the boundary.

At this point, we restrict attention to two and three dimensions. We can construct a hyperbolic manifold by gluing together compact hyperbolic polyhedra, identifying faces in such a way that the hyperbolic structures align and give the total space a coherent hyperbolic structure. From a topological point of view, as in Example 1.1, we only require the links of vertices be spheres.

Now consider the hyperbolic structures on the polyhedra, say by specifying smooth embeddings into $\mathbb{H}^2$ or $\mathbb{H}^3$. Then if two faces can be identified via isometry, this isometry is unique.

Theorem 2.5 (Poincaré's Polyhedron Theorem)**.** *Let X be a space obtained by gluing of n-dimensional compact hyperbolic polyhedra. X is a hyperbolic manifold if and only if the links of vertices are spheres and around each codimension-2 face, the angles formed by codimension-1 faces sum to 2π.*

We can also construct three-manifolds by gluing both finite and ideal polyhedra, which have one or more vertices at infinity. The conditions for a manifold are weakened slightly, as the ideal vertices in the resulting space need not have spherical links. We also have more choice for the gluing maps. If e and e' are identified edges with two ideal vertices, there are $\mathbb{R}$-many choices of identification maps; identifying each edge with $\mathbb{R}$, the gluing map can be any translation $\mathbb{R} \to \mathbb{R}$.

Hyperbolic manifolds can be given a geometric notion of completeness. Suppose M^n is a path-connected hyperbolic manifold. Fixing basepoints, any path in M can be identified with a path in hyperbolic space, with homotopic paths giving homotopic images. By identifying the univeral cover $\tilde{M}$ as the space of homotopy classes of based paths in M, this defines a map $D : \tilde{M} \to \mathbb{H}^n$, the *developing map.*

Definition 2.6. A hyperbolic manifold M is *complete* if the associated developing map $D : \tilde{M} \to \mathbb{H}^n$ is a covering map.

The developing map restricts to a map from $\pi_1(M)$ to $\mathbb{H}^n$, the *holonomy* of M, and the image Γ is the *holonomy group* of M. If M is complete, it is determined by its holonomy: $M = \Gamma\backslash\mathbb{H}^n$. To check a manifold M constructed by gluing ideal polyhedra is complete, it suffices again to consider the behavior at the glued codimension-2 faces, as the behavior of the fundamental group of a manifold lives entirely in codimension 2.

Proposition 2.7. *A manifold is complete if and only if it is metrically complete.*

Proof. (Sketch) The Hopf-Rinow theorem of Riemannian geometry asserts that metric completeness is equivalent to geodesic completeness. Geodesics of M are images of geodesics of $\tilde{M}$. If $D : \tilde{M} \to \mathbb{H}^n$ is a covering map, geodesics of $\mathbb{H}^n$ lift to $\tilde{M}$ and then project to geodesics of M. On the other hand, if M (and therefore $\tilde{M}$) is geodesically complete, the local lifts of $\mathbb{H}^n$ geodesics to $\tilde{M}$ can always be extended, which shows D is a covering map. □

An incomplete manifold can be completed by adding in a geodesic boundary component for each bad ideal vertex. For surfaces, this will always be a circle. In three dimensions, since the boundary components are homeomorphic to the links corresponding ideal vertices, the boundary surfaces may have positive genus.

Thus an incomplete manifold has a compact completion. A noncompact complete 3-manifold of finite volume can also be compactified: the manifold has torus cusps which can be filled via Dehn fillings. *A priori*, there is no reason to believe the resulting manifold might have a hyperbolic structure, but Thurston showed this is the generic case.

Theorem 2.8 (Hyperbolic Dehn surgery)**.** *If M is complete hyperbolic with a torus cusp, all but finitely many Dehn fillings have hyperbolic structures, and these structures are "close" to the complete structure.*

Hyperbolic Dehn surgery illustrates the flexibility in the way a hyperbolic 3-manifold can be compactified. This flexibility is limited, however: the follow result of Mostow shows a given compact hyperbolic 3-manifold admits a unique hyperbolic structure.

Theorem 2.9 (Mostow Rigidity)**.** *Any homotopy equivalence between hyperbolic manifolds of dimension at least 3 is homotopic to an isometry.*

3. Groups acting on trees

In this section we introduce the basics of Bass-Serre theory, the theory of groups acting on trees. This theory provides technical tools for studying 3-manifolds, exploiting the connection between 3-manifolds and finitely generated groups (as their fundamental groups) to translate group theoretic information into concrete topological statements. This follows Serre's treatment in [4].

3.1. **Decompositions of groups and Seifert-van Kampen.** Suppose X and Y are path-connected spaces glued together along a non-empty, path-connected subspace $Z = X \cap Y$. Seifert-van Kampen gives an algebraic description of the fundamental group of $X \cup Y$: it is the amalgamated product $\pi_1(X) *_{\pi_1(Z)} \pi_1(Y)$. Analogously, if X is glued to itself along two path-connected subspaces Y_1 and Y_2, the fundamental group of the resulting space X' decomposes as an HNN decomposition $\pi_1(X') = \pi_1(X)*_{\pi_1(Y_1)}$.

On the other hand, any amalgamated product or HNN extension can be realized in this way. If $G = G_1 *_H G_2$, take X, Y, and Z to be a $K(G_1, 1)$, $K(G_2, 1)$, and $K(H, 1)$, respectively, and glue Z to X and Y via mapping cylinders corresponding to the inclusions induced by $H \hookrightarrow G_1, G_2$. The resulting space has fundamental group G. A similar construction works for an HNN extension.

3.2. **Groups acting on trees and amalgams.**

Definition 3.1. A group G acts *minimally* on a tree T if it does not preserve any proper subtree. G acts *without (edge) inversion* if no element takes an edge to itself with opposite orientation.

Proposition 3.2. *G admits a nontrivial decomposition as an amalgated product $A *_B C$ or HNN extension $A*_B$ if and only if G acts minimally and without inversions on a tree T.*

Proof. (Sketch) ($\Rightarrow$) Take the universal cover of the $K(G, 1)$ constructed in Section 3.1, and construct a tree with a vertex for each lift of X and Y and an edge for each lift of Z.

($\Leftarrow$) Since G acts without inversions, the quotient $G\backslash T$ is still a graph. To further simplify, pick an edge $e \in E(T)$, and contract each component of $T - Ge$ to a point. The result is still a tree, and the new quotient is a graph with exactly one edge. Then take A (and C) to be the vertex stabilizer(s) of the action of G on this quotient and B to be the edge stabilizer. □

Definition 3.3. A *graph of groups* (G, X) is a graph X, with a group G_v for each vertex v and a group G_e for each edge e, and with injective homomorphisms $G_e \to G_u, G_v$ for each edge $e = (u, v)$.

For a graph of groups (G,X), we can define its fundamental group. First, let $F(G,X)$ be the group generated by the vertex groups G_v and elements t_e for all edges e, subject to the relations $t_{e^{-1}} = t_e^{-1}$ and, for $g_u \in G_u$, $g_v \in G_v$ images of the same element of G_e, $g_u t_e = t_e g_v$.

Definition 3.4. Let T be a maximal tree of X. Then the *fundamental group* $\pi_1(G,X,T)$ of (G,X) is the quotient of $F(G,X)$ by the normal subgroup generated by edge generators t_e for $e \in E(T)$.

Remark 3.5. This definition does not depend on the choice of T. An equivalent definition for the fundamental group defines elements as words $g_0 t_{e_1} g_1 t_{e_2} \cdots t_{e_n} g_n$, where $e_1 \cdots e_n$ is a based loop in X and each g_i is an element of the corresponding vertex group in the path.

Example 3.6. If X is a tree, we can take $T = X$, and $\pi_1(G,X,T)$ is the amalgamated product of the G_v along the edge groups G_e.

If a group G acts on a tree Y, then the quotient $X = G\backslash Y$ can be given the structure of a graph of groups, by taking as vertex groups the vertex stabilizers of G acting on X and edge homomorphisms corresponding to the inclusions of the edge stabilizers into each vertex stabilizers.

Theorem 3.7. *With identifications as above, $G \cong \pi_1(G,X,T)$.*

3.3. **Property (FA).** Serre's property (FA) gives a characterization of groups which are *not* amalgamated products, in terms of their actions on trees.

Definition 3.8. A group has *property (FA)* if any action of it on a tree has a fixed point.

To check the action of a finitely generated group G has a fixed point, it suffices that each element of the group fix some point, by an analogous argument to Helly's theorem about intersections of convex subsets of $\mathbb{R}^n$. In fact, G has a fixed point if just the generators and their pairwise products each fix a point.

An important class of groups to consider in this context are nilpotent groups. The action of a nilpotent group on a tree is highly constrained.

Proposition 3.9 (Serre)**.** *Let G be a finitely generated nilpotent group acting on a tree X. Then either*

(1) *G fixes a point, or*
(2) *G acts by translations on a bi-infinite path in X.*

In particular, a slightly weaker condition implies a finitely generated nilpotent group has (FA): if each generator of G fixes a point, then the entire group has a fixed point.

Example 3.10. The group $\mathrm{SL}(3,\mathbb{Z})$ has property (FA). While $\mathrm{SL}(3,\mathbb{Z})$ is not nilpotent, it has many large subgroups which are, including ones generated by pairs of generators of $\mathrm{SL}(3,\mathbb{Z})$. Applying the above Proposition 3.9 to these subgroups shows each generator and each product of two generators must always fix a point, so $\mathrm{SL}(3,\mathbb{Z})$ does as well.

Property (FA) is closed related to property (T). Recall that a group G has property (T) if every affine isometric action of G on a Hilbert space has a fixed point. Equivalently, if $\rho : G \to \mathrm{U}(E)$ is any isometric action of G on a Hilbert space E, one has $H^1(G;E) = 0$.

Proposition 3.11. *Property (T) implies property (FA).*

This follows from the fact that any tree embeds canonically and isometrically in some Hilbert space. On the other hand, (T) is inherited by finite index subgroups, whereas (FA) is not, so these are not equivalent.

A compact orientable irreducible 3-manifold is *Haken* if it contains a properly embedded 2-sided essential surface. In this sense, a non-Haken irreducible 3-manifold is small, as all such surfaces are inessential.

Example 3.12. If M is a non-Haken irreducible 3-manifold, then $\pi_1(M)$ has (FA). If M is virtually Haken, then $\pi_1(M)$ has a finite index cover which does not have (FA). So $\pi_1(M)$ does not have (T).

3.4. **Stallings Theorem on ends of groups.**

Definition 3.13. The *ends* of a space X are the elements of the set

$$\mathcal{E}(X) = \varprojlim \pi_0(X - K),$$

the inverse limit taken over all compact subsets $K \subseteq X$.

This can be extended to define the ends of a finitely generated group G, by defining $\mathcal{E}(G) = \mathcal{E}(C_S(G))$, where $C_S(G)$ is the Cayley graph of G with respect to some finite generating set S. This definition is independent of the choice of generating set, as $\mathcal{E}(G) = \mathcal{E}(X)$ for any space X on which G acts properly and cocompactly. The number of ends of G is closely related to to the structure of the group, particularly if G has a large number of ends.

Theorem 3.14 (Stallings Ends Theorem)**.** *Let G be a finitely generated group. Then the number of ends of G is one of $0, 1, 2, \infty$. Moreover,*

(1) *number of ends is 0 iff G is finite;*
(2) *number of ends is 2 iff G is virtually $\mathbb{Z}$;*
(3) *number of ends is ∞ iff G is not virtually $\mathbb{Z}$, but splits as a nontrivial amalgam or HNN extension over a finite group.*

The idea of the proof is to consider "cuts" of (the Cayley graph of) G which coarsely separate, and to define a suitable complexity function on such cuts so that for each cut C of least complexity, and each $g \in G$, either gC is disjoint from, or equal to C. In particular, the decomposition of G by such a system of cuts is encoded combinatorially as a tree with a natural G action. This action is minimal and without inversions, so Proposition 3.2 gives the appropriate decomposition.

3.5. **Fields with discrete valuations.** This theory gives a method for understanding the structure of the group $\mathrm{SL}(2, K)$. This group arises naturally in the study of geometric structures on 3-manifolds. For a hyperbolic 3-manifold M, we will find ways of realizing the fundamental group of M as a discrete subgroup of $\mathrm{SL}(2, K)$ for some number field K.

A *discrete valuation* on a field K is a map $\nu : K \to \mathbb{Z} \cup \{\infty\}$ satisfying (1) $\nu(xy) = \nu(x) + \nu(y)$; (2) $\nu(x + y) \geq \min(\nu(x), \nu(y))$; and (3) $\nu(x) = \infty$ if and only if $x = 0$. The *valuation ring* corresponding to ν consists of all field elements with nonnegative valuation.

Example 3.15. The field $\mathbb{Q}$ has a p-adic valuation for every (rational) prime p. The valuation ring consists of rationals with denominator coprime to p. The field $\mathbb{C}(t)$ (i.e. the field of rational functions on the Riemann sphere) has a valuation for every point on the Riemann sphere, whose valuation ring consists of rational functions with no pole at that point.

If K is a field with valuation ring R and maximal ideal m and quotient field $k = R/m$, and K has completion K_m with respect to the valuation, then $\mathrm{SL}(2, K)$ acts on a tree so that vertex links are copies of the projective line over k, and the space of ends of the tree is the projective line over K_m. The point stabilizers are conjugates of $\mathrm{SL}(2, R)$.

Example 3.16. Suppose M is a hyperbolic 3-manifold. There is a representation $\rho : \pi_1(M) \to \mathrm{PSL}(2, \mathbb{C})$ which comes from its hyperbolic structure. By Mostow rigidity, this representation is conjugate into $\mathrm{PSL}(2, K)$ for some number field K, which lifts to $\mathrm{SL}(2, K)$ (because orientable 3-manifolds are parallelizable). Then either the image is conjugate into $\mathrm{SL}(2, A)$ for some ring of algebraic integers, or M is Haken.

Example 3.17. Let M be a hyperbolic 3-manifold with a cusp. By Thurston, there is a 1 (complex) dimensional space of representations to $\mathrm{SL}(2, \mathbb{C})$ deforming the complete structure. Let C be the complex 1-dimensional variety of such deformations, and let K denote the function field of C. Then there is a representation $\pi_1(M) \to \mathrm{SL}(2, K)$, and ideal points on C correspond to discrete valuations on K for which the action of $\pi_1(M)$ on the associated tree describes the "limit" of the actions on $\mathbb{H}^3$ corresponding to ordinary points on C.

4. Twisted homology and the Thurston norm

Up to this point, we have primarily focused on classical results in 3-manifolds. In this section we introduce more recent methods for studying 3-manifolds, which rely on using homology with twisted coefficients to capture information about surfaces embedded in our manifold.

4.1. **Genus of a knot and Thurston norm.** Let K be a knot in the 3-sphere. A compact oriented embedded surface S in the 3-sphere with boundary K is a *Seifert surface*. The *genus* of K is the least genus of a Seifert surface, denoted $g(K)$.

Knot genus is closely related to a more general topological measure of complexity in 3-manifolds, the Thurston norm.

If M is an irreducible, atoroidal 3-manifold, there is a norm $||\cdot||$ on $H_2(M;\mathbb{R})$, the *Thurston norm*, introduced in [5]. For a homology class $\alpha \in H_2(M;\mathbb{Z})$, $||\alpha||$ measures the minimal complexity of an embedded surface representing α. Precisely, it is defined as

$$||\alpha|| = \inf_{[S]=\alpha} \sum_{S_i \subseteq S} \max\{0, -\chi(S_i)\}.$$

This norm is convex and, when restricted to rays through the origin, linear. Thus this definition can be extended to a norm on $H_2(M;\mathbb{R})$.

Theorem 4.1 (Thurston). *The unit ball of the Thurston norm is a rational polyhedron with integral lattice vertices.*

This partitions $H_2(M;\mathbb{R})$ into cones, where a cone corresponding to a top dimensional face of the unit ball consists of all points on rays which intersect the face.

Theorem 4.2 (Thurston). *If M fibers over S^1, the homology class of the fiber lies in the interior of a cone. Moreover, any other homology class in $H_2(M;\mathbb{Z})$ in the same cone is realized as the fiber of a fibration $M \to S^1$.*

These are the *fibered faces* of the unit ball. A recent theorem of Agol shows every closed, hyperbolic 3-manifold has a finite cover which fibers over the circle.

For a knot $K \subseteq S^3$, the genus of K is related to the Thurston norm of the homology class α dual to a generator of $H^1(S^3 - K, \mathbb{Z})$ by

$$||\alpha|| = 2g(K) - 1.$$

In practice, the Thurston norm can be difficult to compute. However, producing a (not necessarily minimal) Seifert surface associated to K gives an upper bound on the Thurston norm of the corresponding homology class.

4.2. **Alexander polynomial.** In this section we introduce another knot invariant, the Alexander polynomial. If K is a knot in the 3-sphere, then its complement M has $H_1(M;Z) = \mathbb{Z}$, so there is an infinite cyclic cover $\hat{M}$ with fundamental group equal to the commutator subgroup of $\pi_1(M)$. The first homology of $\hat{M}$ is a $\mathbb{Z}[t, t^{-1}]$-module, where t acts generating the deck group of the cover.

In fact, it is a *cyclic* module, with a presentation of the form $\mathbb{Z}[t,t^{-1}]/\langle f(t)\rangle$ for some polynomial $f(t)$, unique up to multiplication by a unit. This is the Alexander polynomial (usually normalized to have nonzero constant term).

The module, and this polynomial, can be explicitly calculated from a Seifert surface, by computing the *Seifert matrix*. This calculation shows twice the genus of any Seifert surface is an upper bound for the degree of the Alexander polynomial. In particular, this gives a lower bound on knot genus,

$$g(K) \geq \frac{1}{2} \deg f,$$

and so also a lower bound on the Thurston norm.

For certain classes of knots, e.g. fibered knots, this bound is an actual equality. However, in general, the Alexander polynomial of a knot does not determine its genus.

4.3. **Homology with twisted coefficients.** Our goal is to find an improvement of the Alexander polynomial. This improvement is the *twisted Alexander polynomial*, which we define analogously, except replacing the usual homology with homology with coefficients twisted by a representation $\alpha : \pi_1(S^3 - K) \to \mathrm{GL}(V)$ to a finite dimensional k-vector space V. In this section, we review the basics of homology and cohomology with twisted coefficients.

For a space M with fundamental group $\pi = \pi_1(M)$, universal cover $\tilde{M}$, and a $\mathbb{Z}[\pi]$-module E, the homology groups $H_*(M;E)$ are defined by the chain groups $C_n(M;E) = C_n(\tilde{M}) \otimes_{\mathbb{Z}[\pi]} E$, with boundary maps $(\partial_n \otimes \mathrm{id})$.

A corresponding definition can be given for cohomology with E-coefficients; the cochain groups are defined as the group of $\mathbb{Z}[\pi]$-homomorphisms $C^n(M;E) = \mathrm{Hom}_{\mathbb{Z}[\pi]}(C_n(\tilde{M}), E)$. The coboundary map is defined in the same way as for usual cohomology.

Example 4.3. When E is a trivial $\mathbb{Z}[\pi]$-module, the (co)-chain groups (and therefore the (co)-homology groups) are isomorphic to the usual groups.

Example 4.4. If $E = \mathbb{Z}[\pi]$ with the natural action, then $H_*(M;E) = H_*(\tilde{M})$.

Example 4.5. For a manifold M, its zero dimensional (co)-homology groups with coefficients in a $\mathbb{Z}[\pi]$-module E can be identified as follows:

(1) $H^0(M;E) = \{v \in E : gv = v \ \forall g \in \pi_1(M)\} = E^\pi$, the $\pi_1(M)$ invariants of E.
(2) $H_0(M;E) = E/\{gv - v : g \in \pi_1(M),\, v \in E\} = E_\pi$, the $\pi_1(M)$ co-invariants of E.

Example 4.6. The first cohomology group of a manifold M with coefficients in E are the *twisted homomorphisms* $f : \mathbb{Z}[\pi] \to E$, where

$$f(gh) = f(g) + gf(h),$$

modulo the homomorphisms defined by $\tilde{x}(g) = gx - x$ for each $x \in E$.

An alternate but equivalent definition of (co)-homology with twisted coefficients can be stated by using the $\mathbb{Z}[\pi]$-module structure on E to define a bundle over M with fiber E. The chain groups consist of finite sums of n-cells with coefficients not in E, but lifts to the bundle over M, additionally interpreting a coefficient valued 0 in E to actually be 0.

Many familiar theorems from algebraic topology still hold with twisted coefficients. In particular, Poincaré duality still holds for closed, oriented manifolds.

In the context above, where we use twisted homology to define the twisted Alexander polynomial, E is a vector space V, and the choice of a $\mathbb{Z}[\pi]$-module structure on V is equivalent to picking a representation $\alpha : \pi_1(M) \to \mathrm{GL}(V)$. By considering the extra algebraic information given by the representation α, the resulting polynomial gives better topological information about K, and provides a better bound on the knot genus.

4.4. **Sutured manifolds and tautness.** A compact oriented 3-manifold with boundary $(M, \partial M)$ with a decomposition $\partial M = R_+ \cup_\gamma R_-$, γ a collection of oriented, simple closed curves in ∂M, is said to be *sutured.* A sutured manifold is moreover *taut* if it is irreducible and both components $R_\pm$ are incompressible and achieve the Thurston norms for their respective homology classes.

Example 4.7. A knot complement cut open along a minimal genus Seifert surface is a taut sutured manifold.

An irreducible sutured manifold is *balanced* if $\chi(R_+) = \chi(R_-)$, M is not a solid torus with no sutures, and all components of each of $R_\pm$ have non-positive Euler characteristic unless $M = D^3$ with γ a single curve.

Given a module E, a sutured manifold M is an *E-homology product* if the maps $H_*(R_\pm;E) \to H_*(M;E)$ induced by the natural inclusions are isomorphisms. When E comes from a representation $\alpha : \pi_1(M) \to \mathrm{GL}(V)$, we also say M is an α-homology product.

Theorem 4.8 (Friedl and Kim, [3])**.** *Suppose M is an irreducible sutured manifold with no component a solid torus without sutures. Then if M is an E-homology product, M is taut.*

Proof. (Sketch) The conditions on M ensure any torus components of $R_\pm$ are incompressible, so it suffices to show $R_\pm$ are minimal genus within their homology class. Take S a surface in $[R_\pm]$ which separates M into two pieces, labelled $M_\pm$. Homological arguments show the homology groups $H_*(S;E)$ surject onto $H_*(M_\pm;E)$ and likewise $H_*(R_\pm;E)$ inject into $H_*(M_\pm;E)$, with isomorphisms for $* \neq 1$. Thus $\chi(S) \geq \chi(R_\pm)$. ☐

This gives a way to verify M is taut. In particular, if M is obtained by cutting a knot complement open along a Seifert surface S, this gives a method to certify that S is minimal by producing a representation $\alpha : \pi_1(M) \to \mathrm{GL}(V)$ for which M is an α-homology product. Friedl and Kim also show that under additional certain assumptions, the twisted Alexander polynomial associated to α gives a *sharp* bound on the Thurston norm of $[S]$.

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