Surfaces in 3-manifolds and the Thurston norm

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Fields Institute
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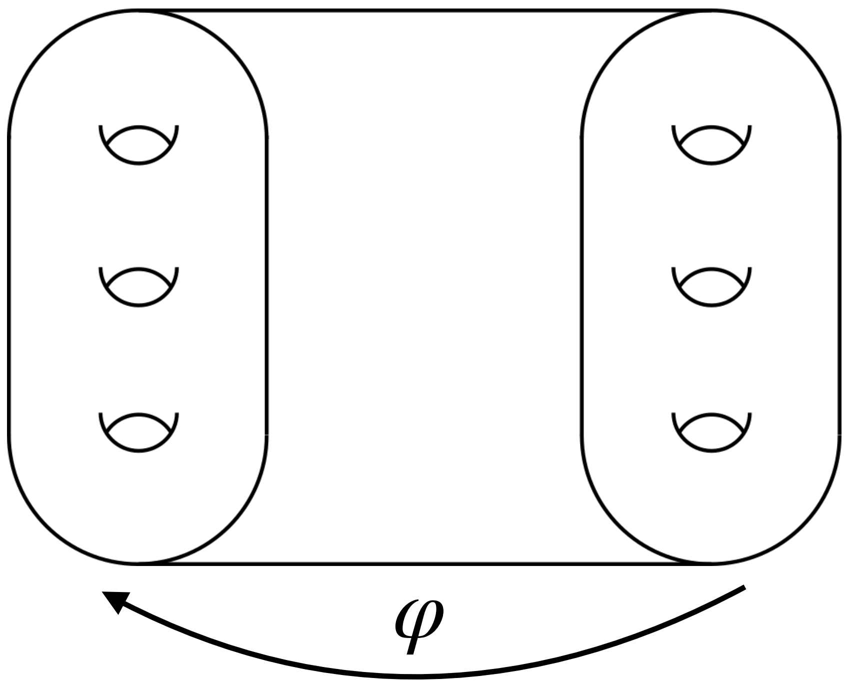

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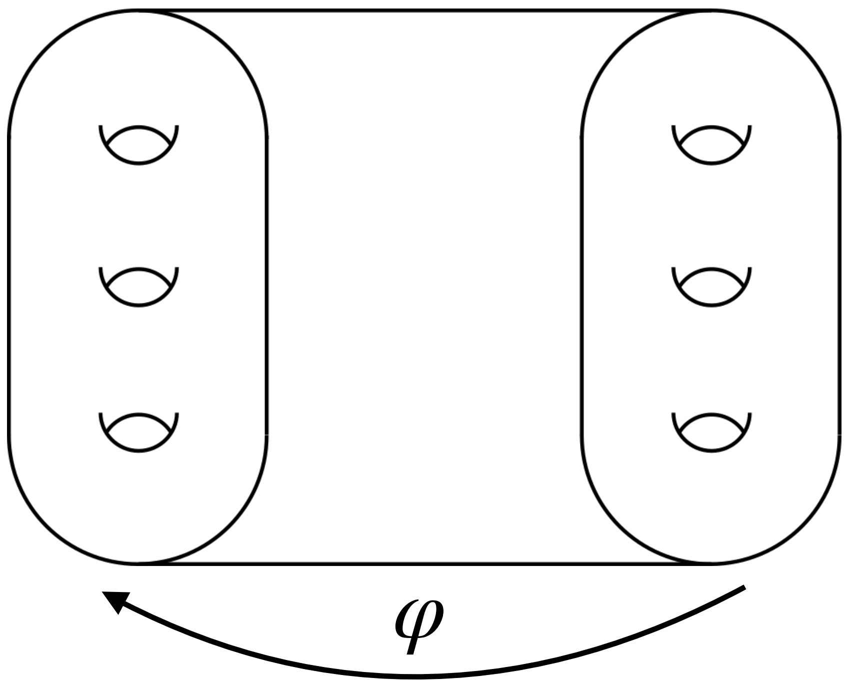

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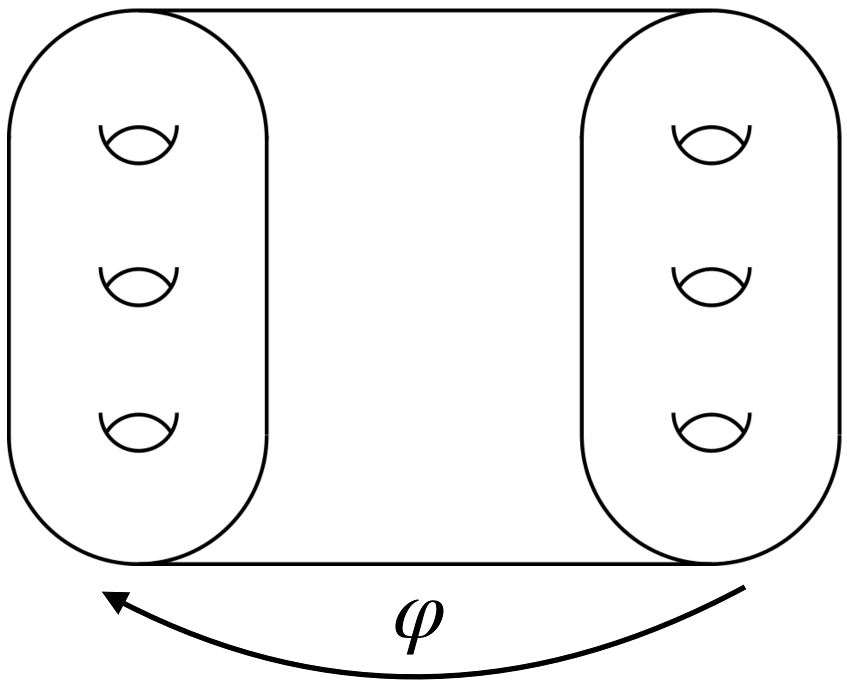

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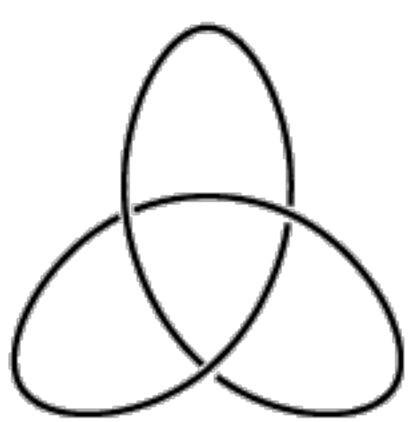

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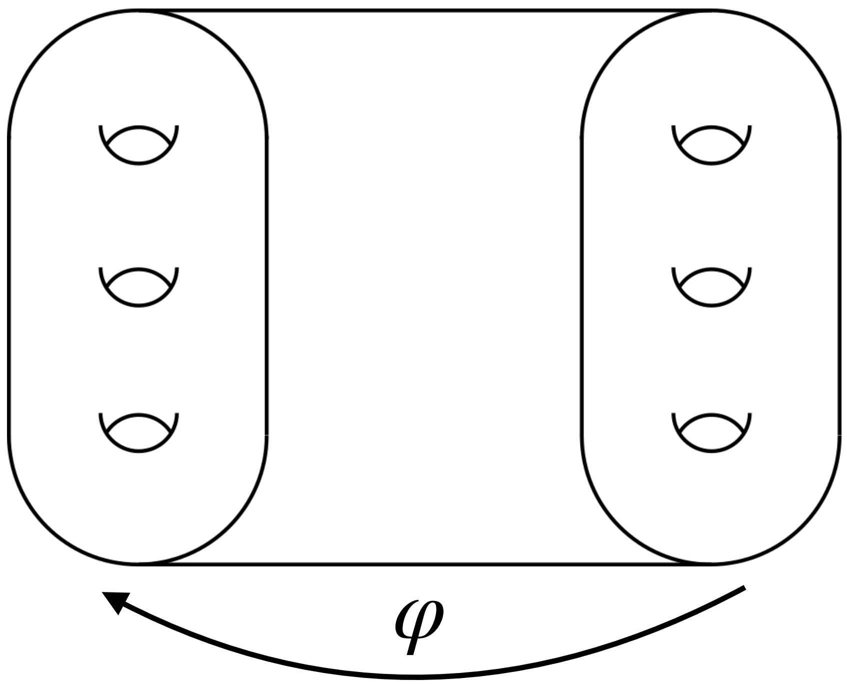

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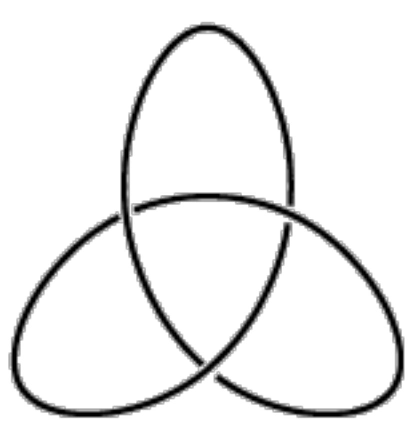

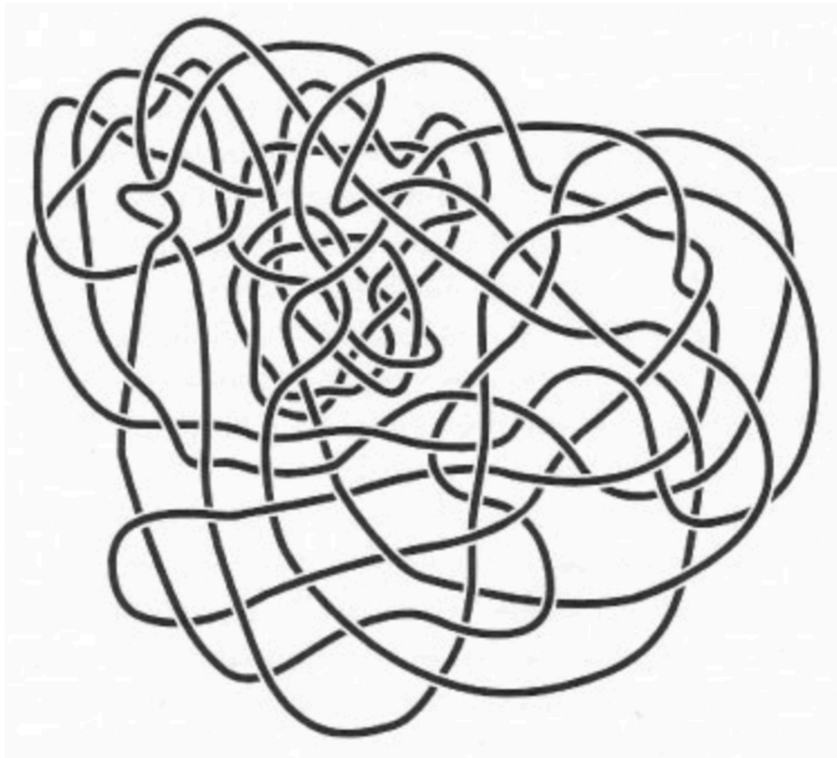

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  - Kneser's lemma: $\pi_1(S) \to \pi_1(M)$ kernel $\Rightarrow$ embedded disk (compressing disk)

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$$[\mathbb{R}^3, \mathbb{S}^3, \mathbb{H}^3, \mathbb{S}^2 \times \mathbb{R}, \mathbb{H}^2 \times \mathbb{R}, \text{Sol}, \text{Nil}, \widetilde{\text{SL}_2(\mathbb{R})}]$$

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M_φ is a fibered 3-manifold: $S \to M_\varphi \to S^1$

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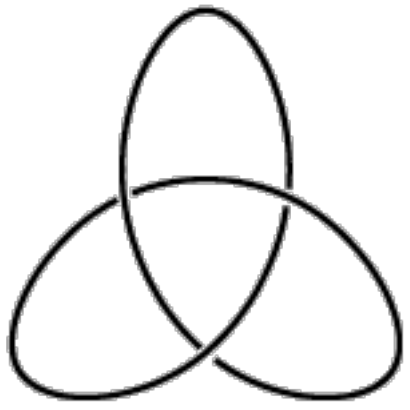

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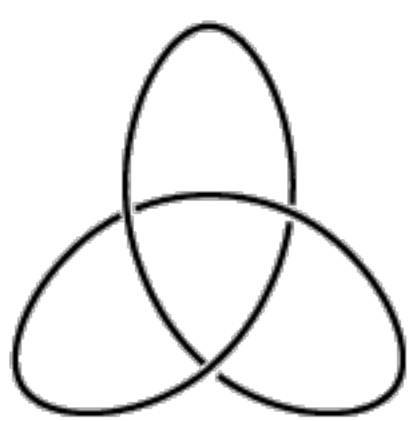

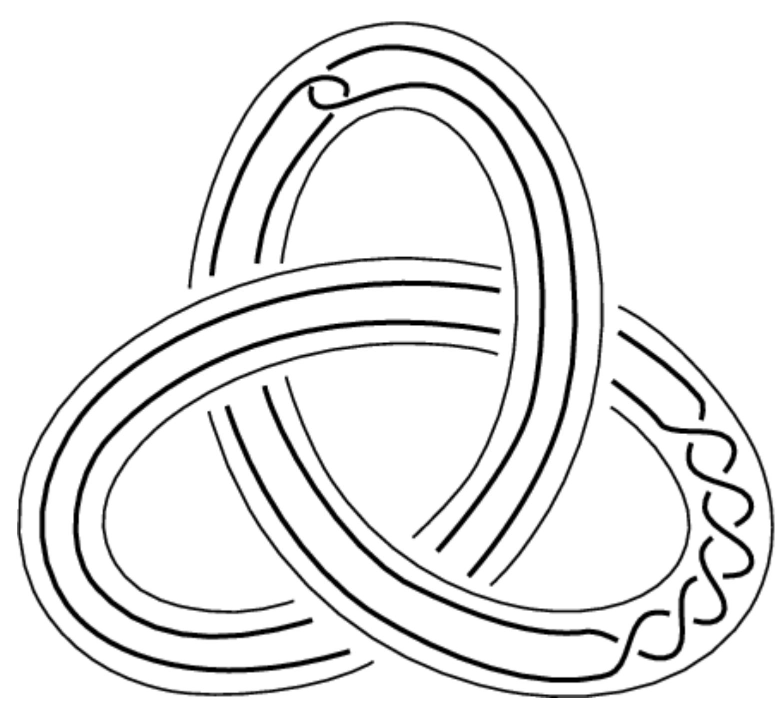

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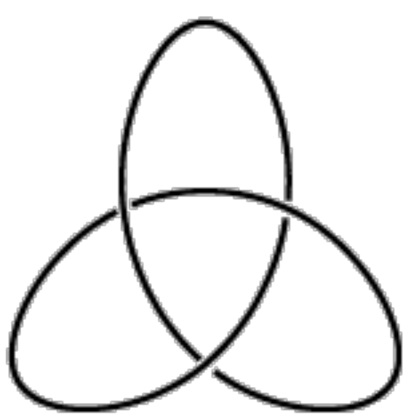

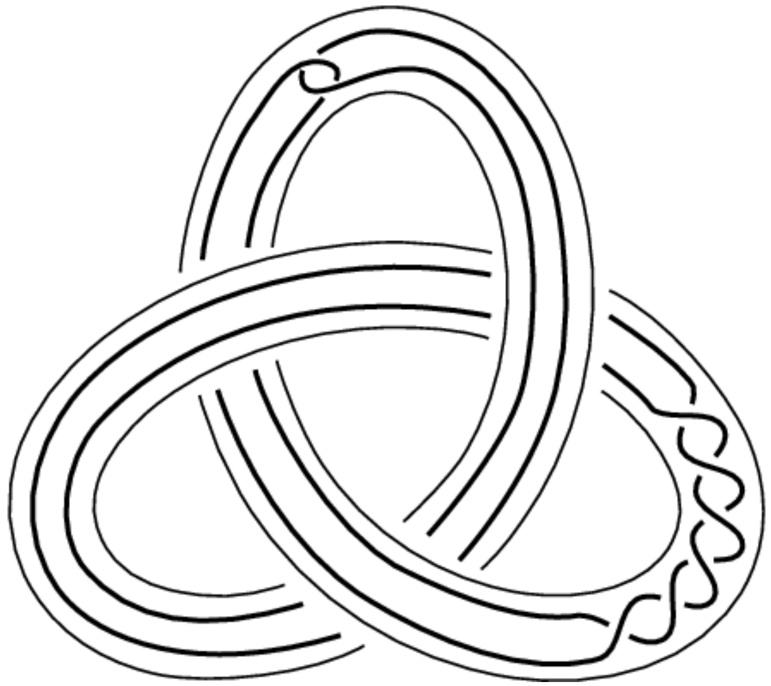

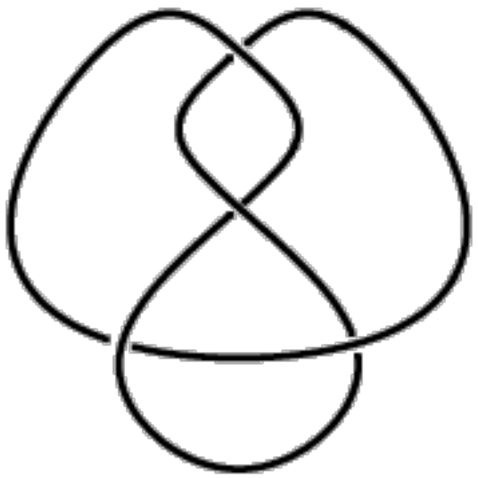

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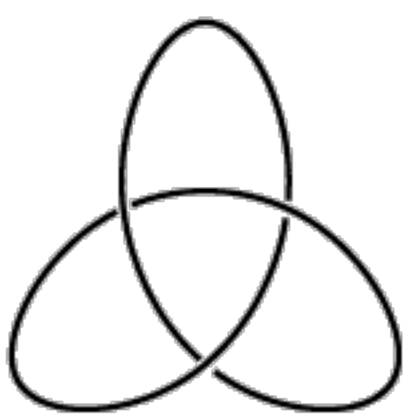

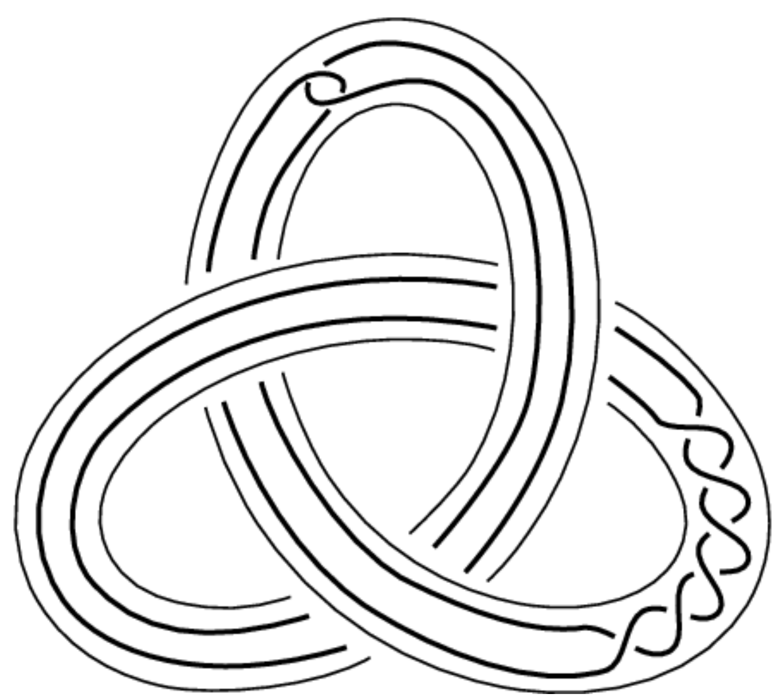

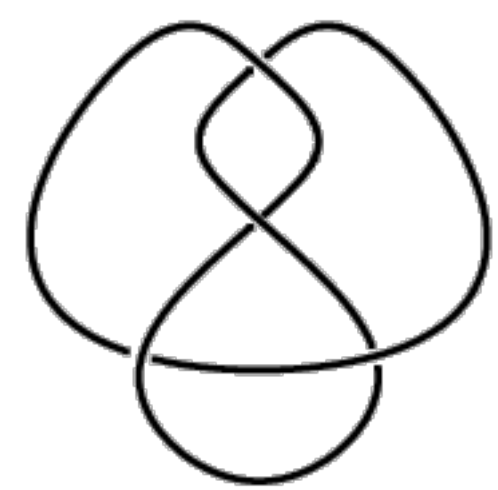

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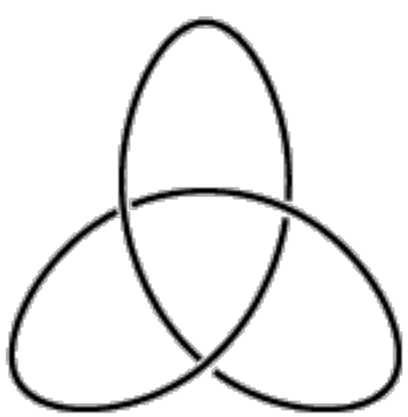

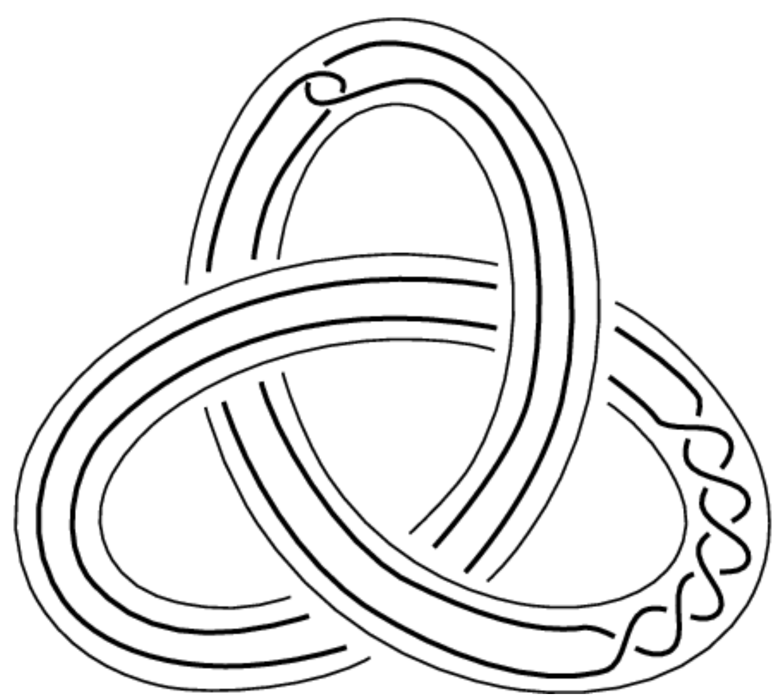

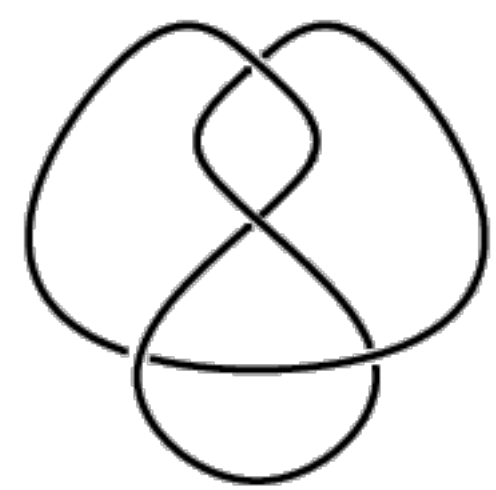

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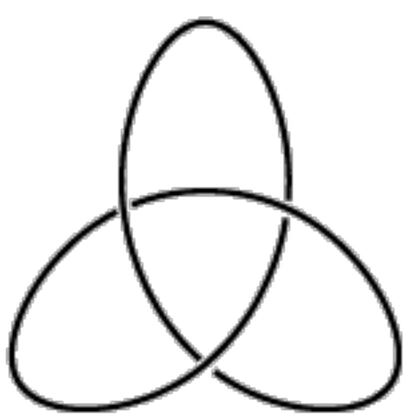
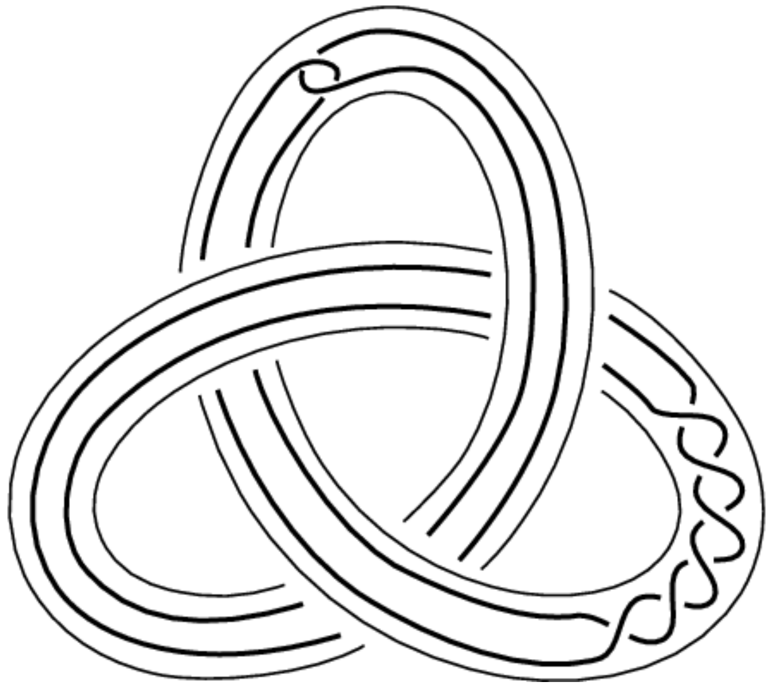
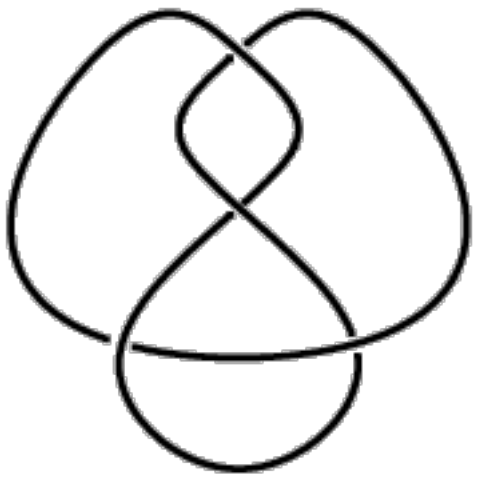

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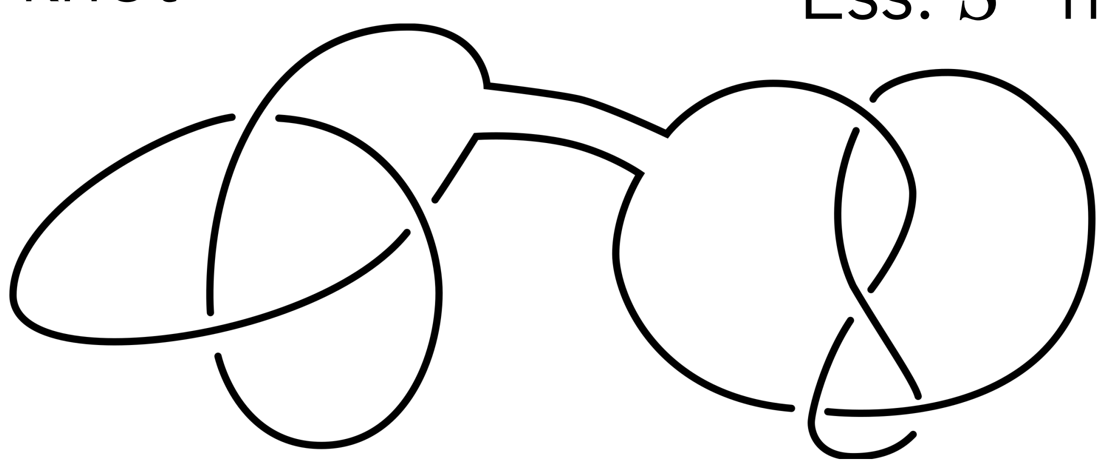

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Defn. The knot genus $g(K)$ is the minimum genus among Seifert surfaces of K.

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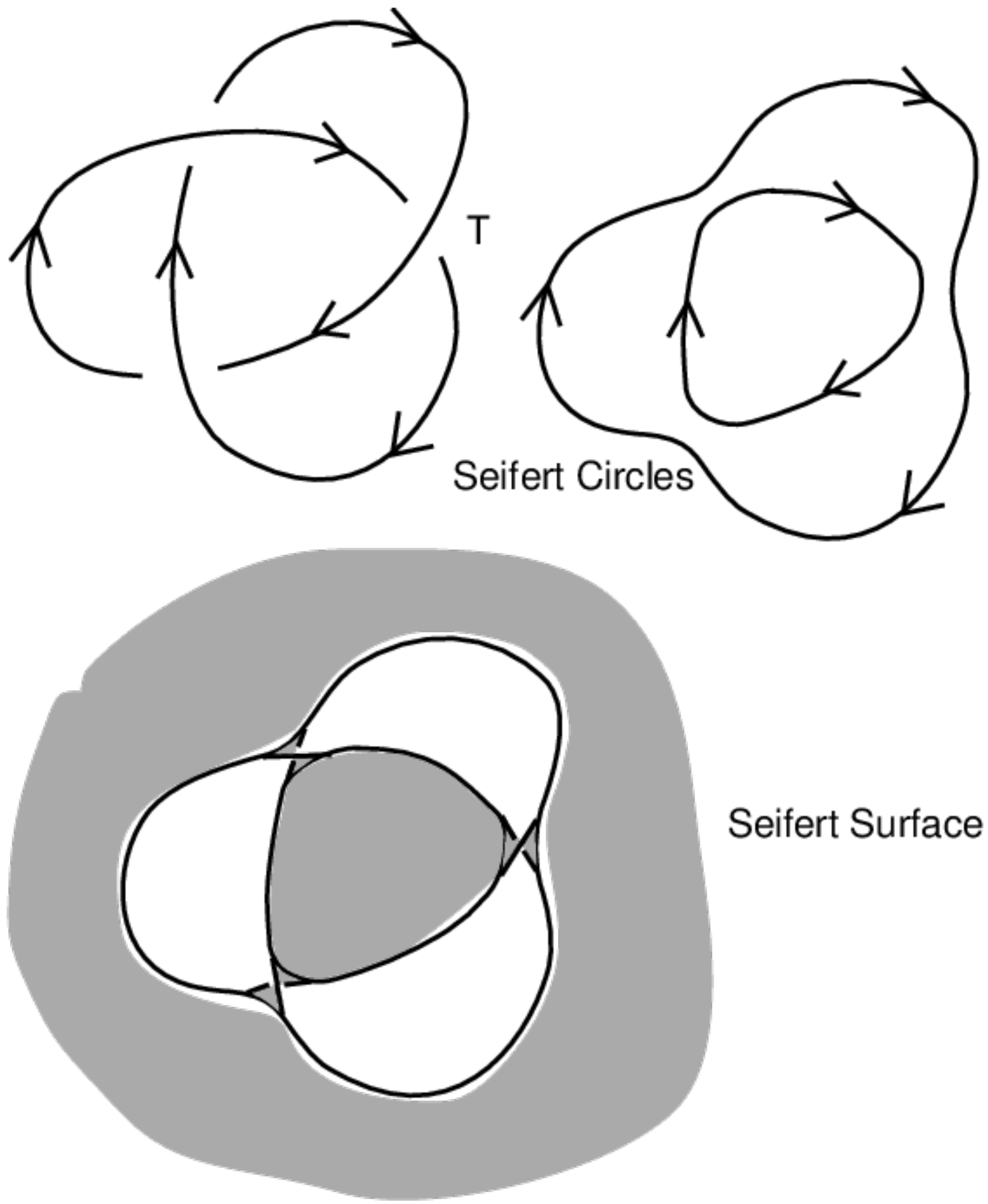

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Nb. All Seifert surfaces are homologous!

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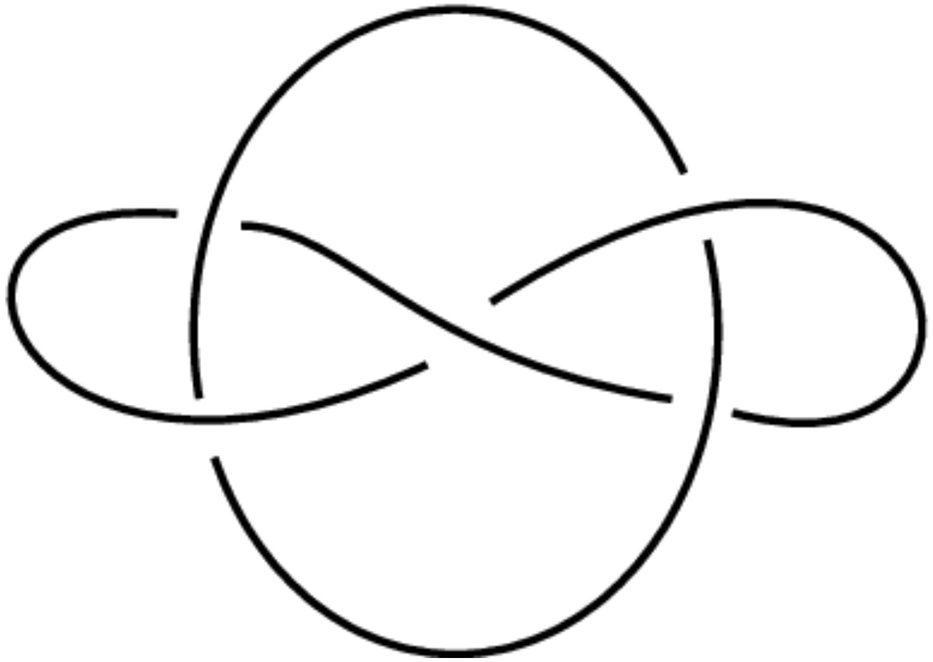
Example: $S \subset S^3 - K$ a Seifert surface

$$\|[S]\| = 2g(K) - 1$$

Thurston norm

Example: Whitehead link



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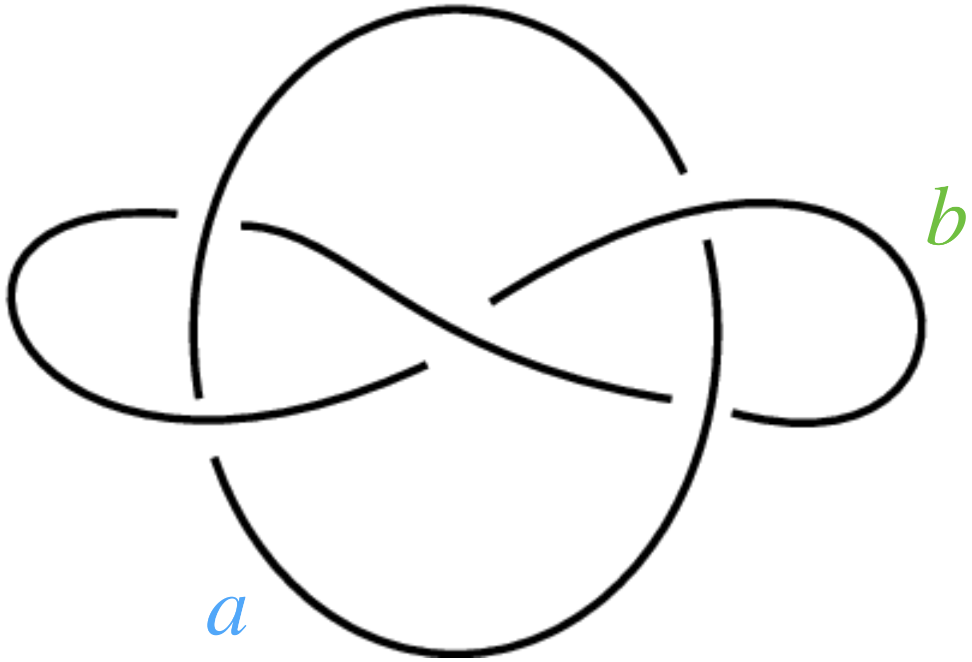

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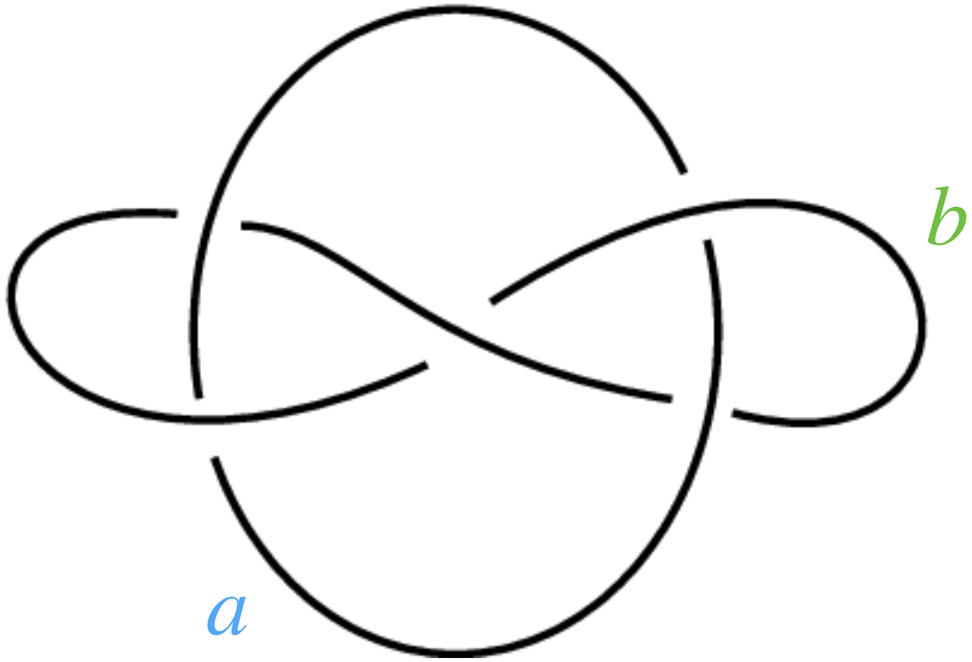

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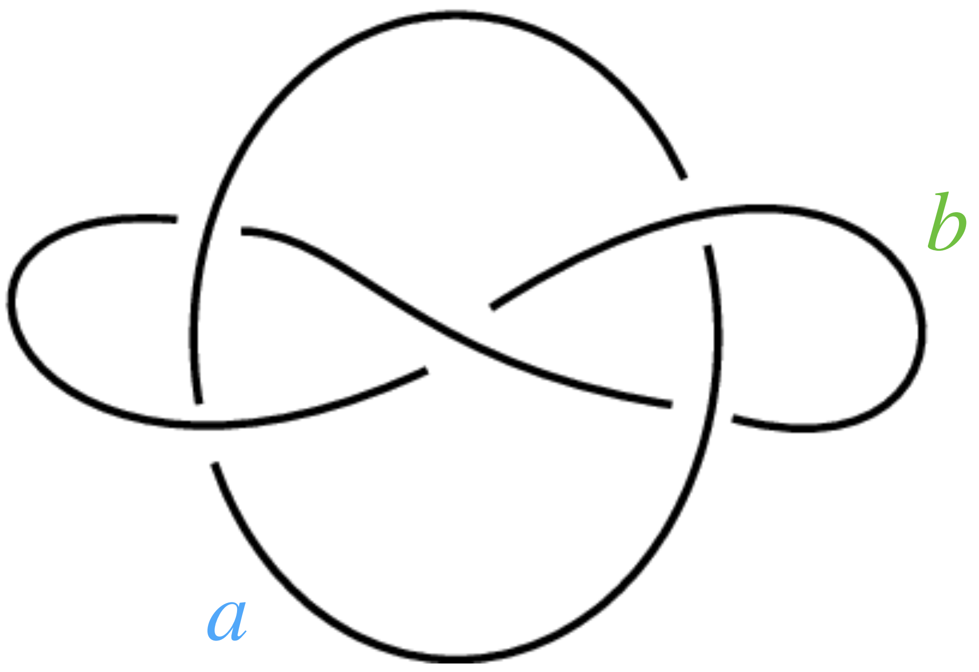
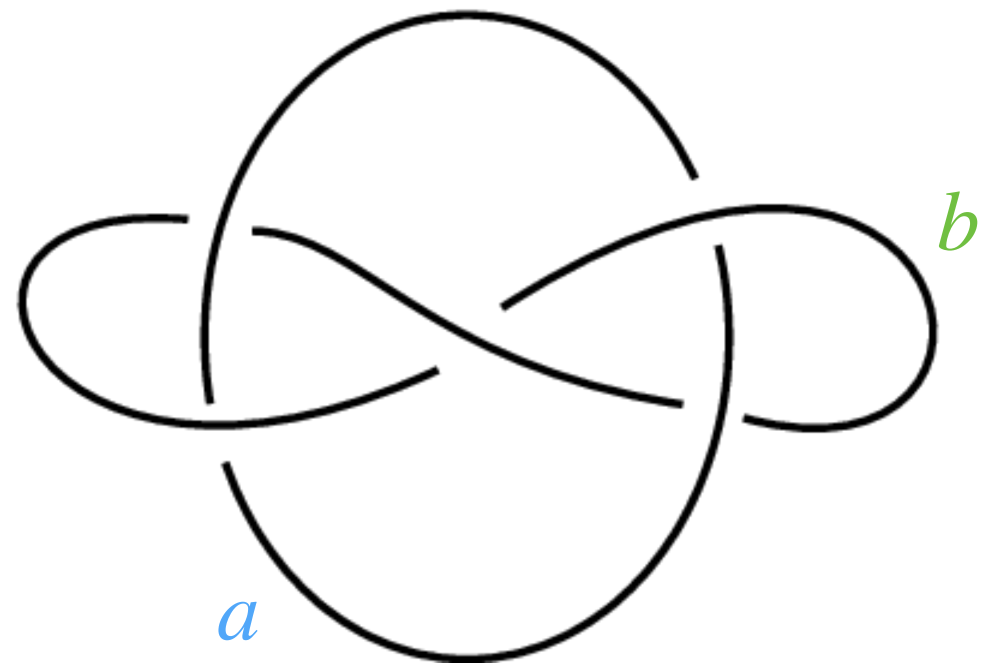
What about $\|3a + 16b\|$?



Thurston norm

Example: Whitehead link



$$H_2(S^3 - L) \cong \mathbb{Z}^2 = \langle a, b \rangle$$

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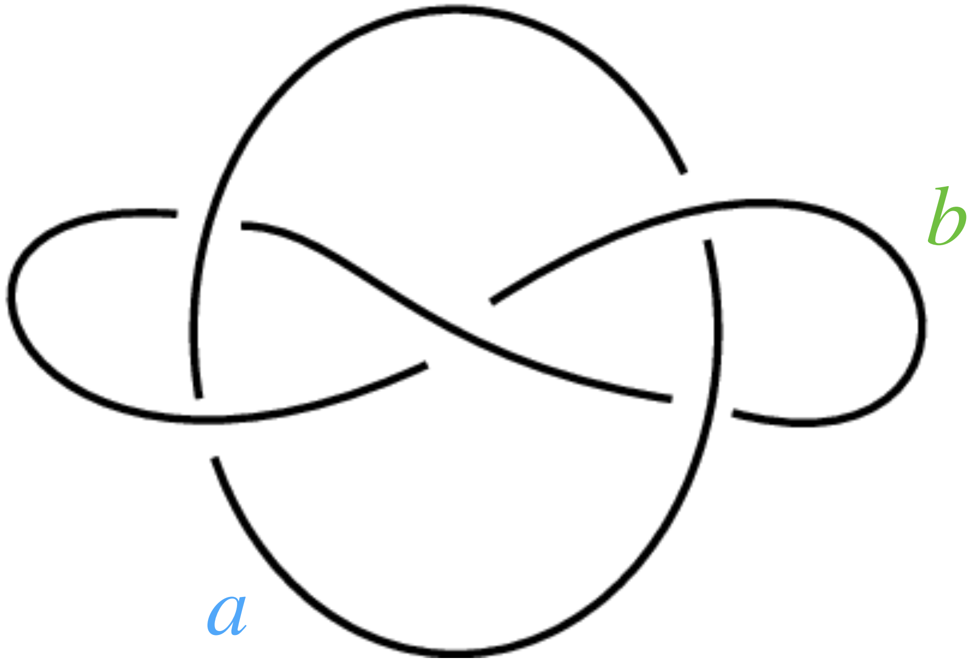

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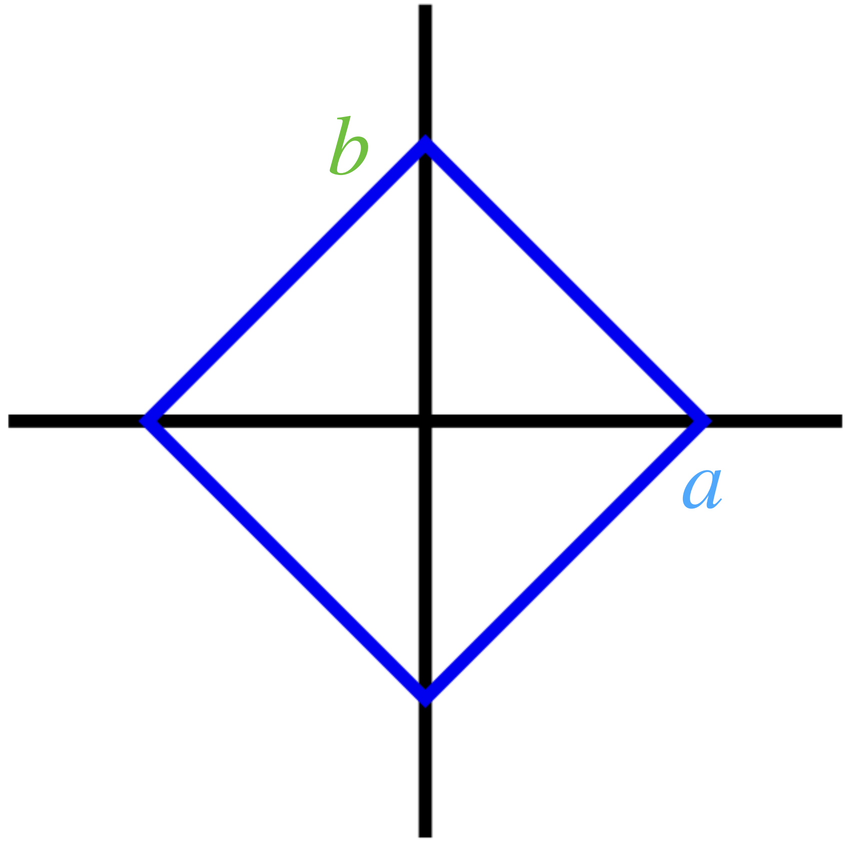

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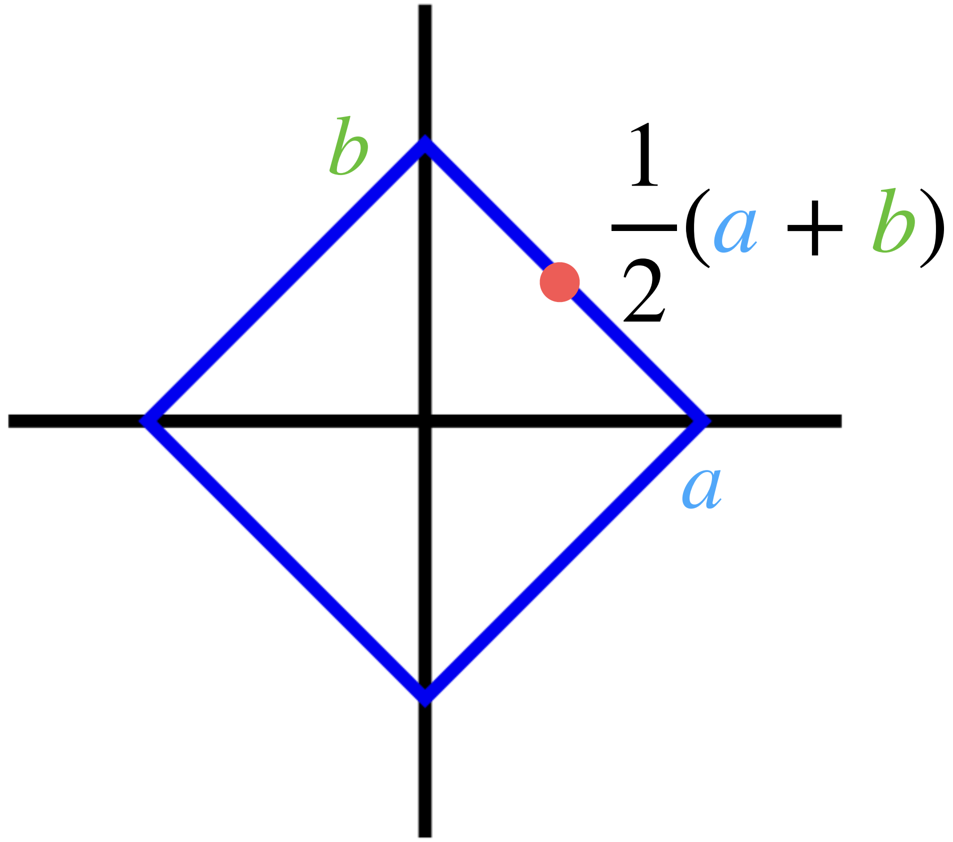

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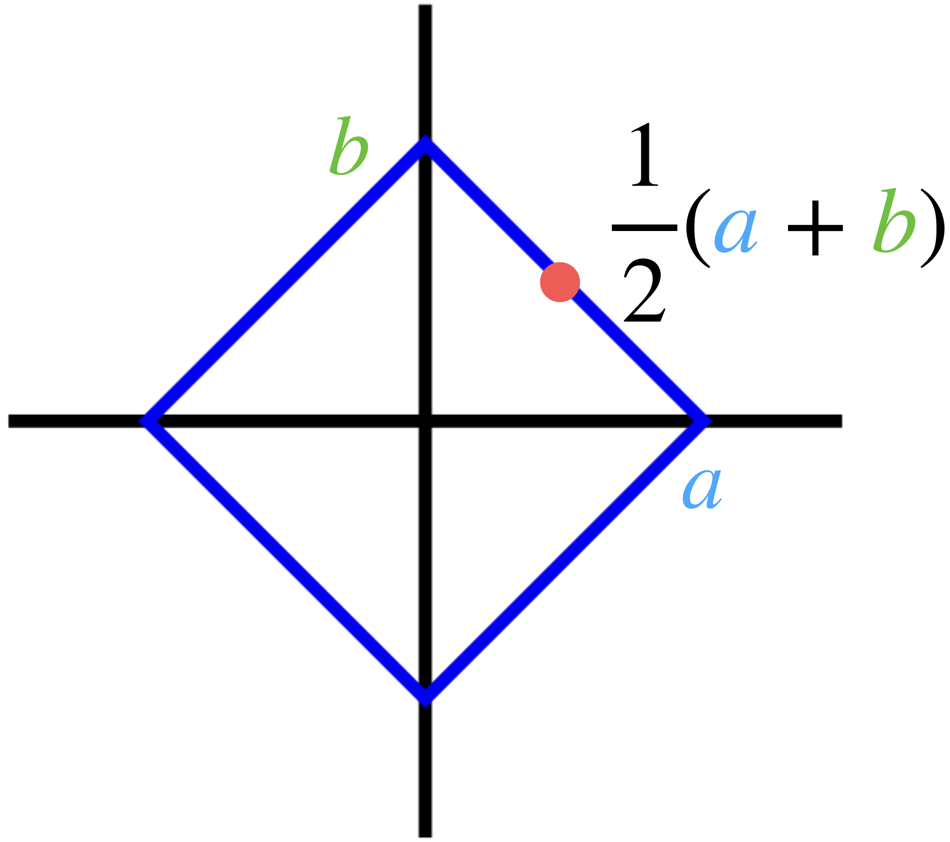

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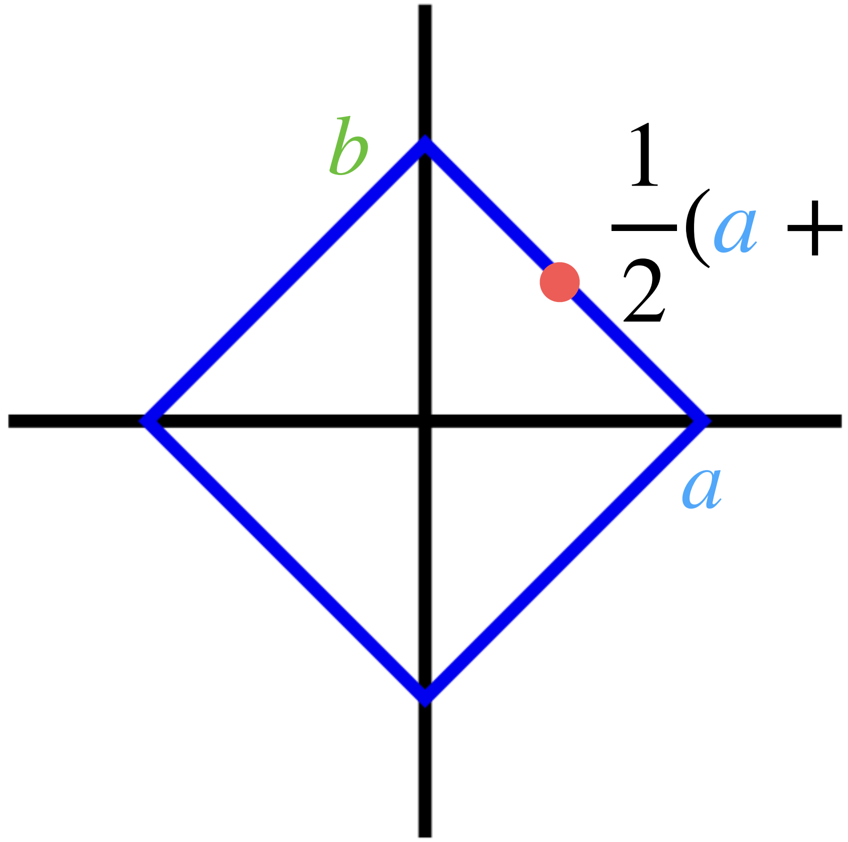

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Thurston norm

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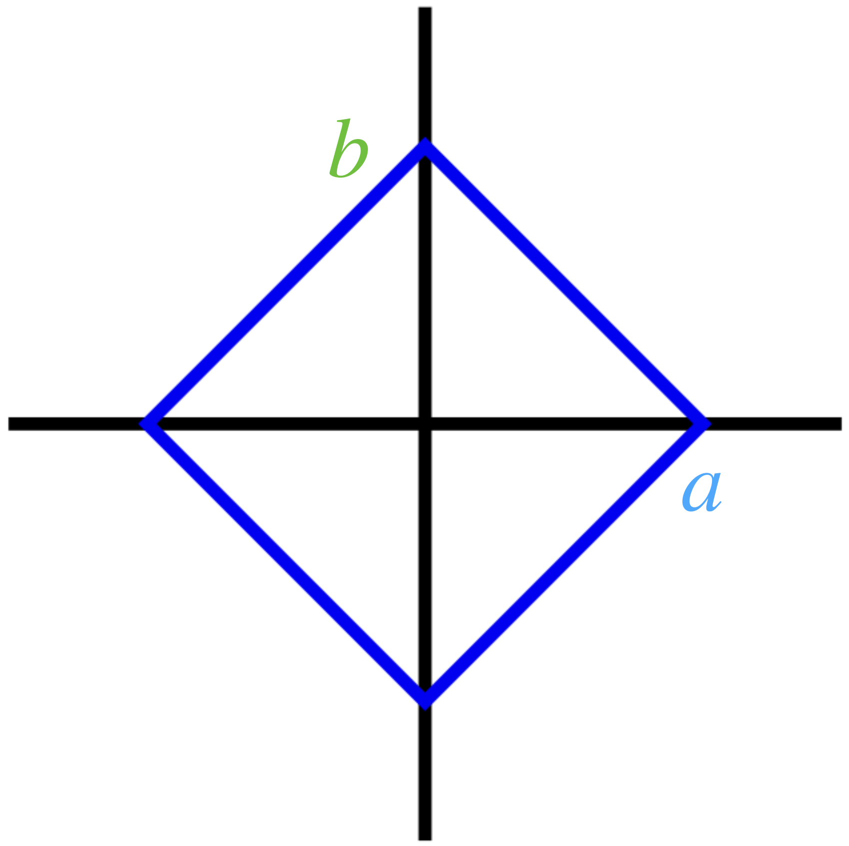
$$S \to M \to S^1$$



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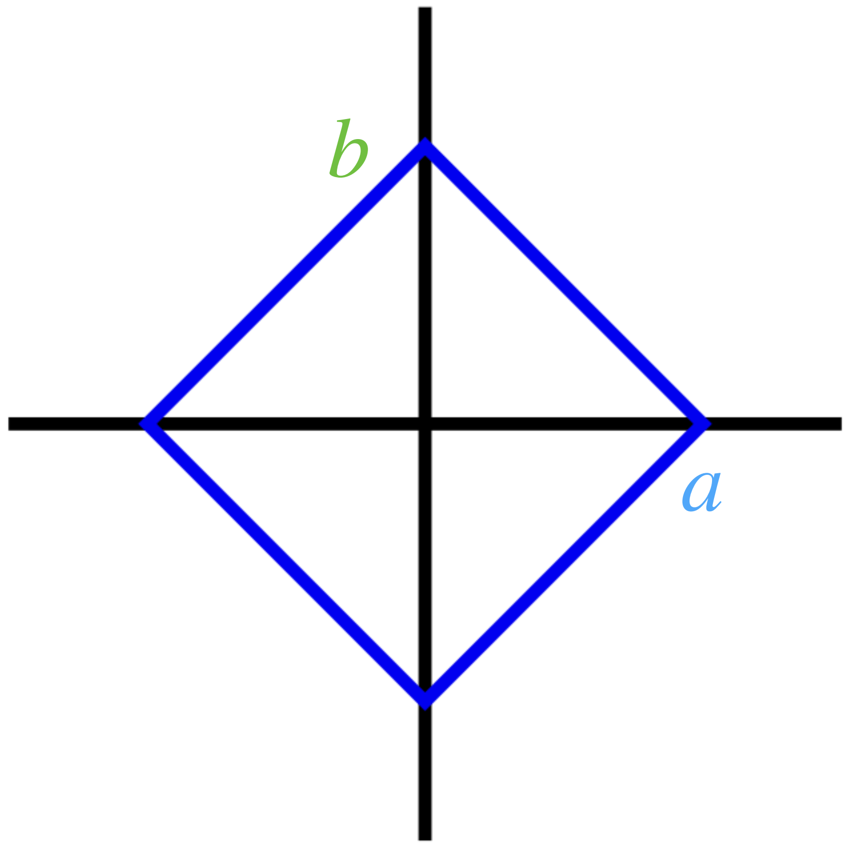

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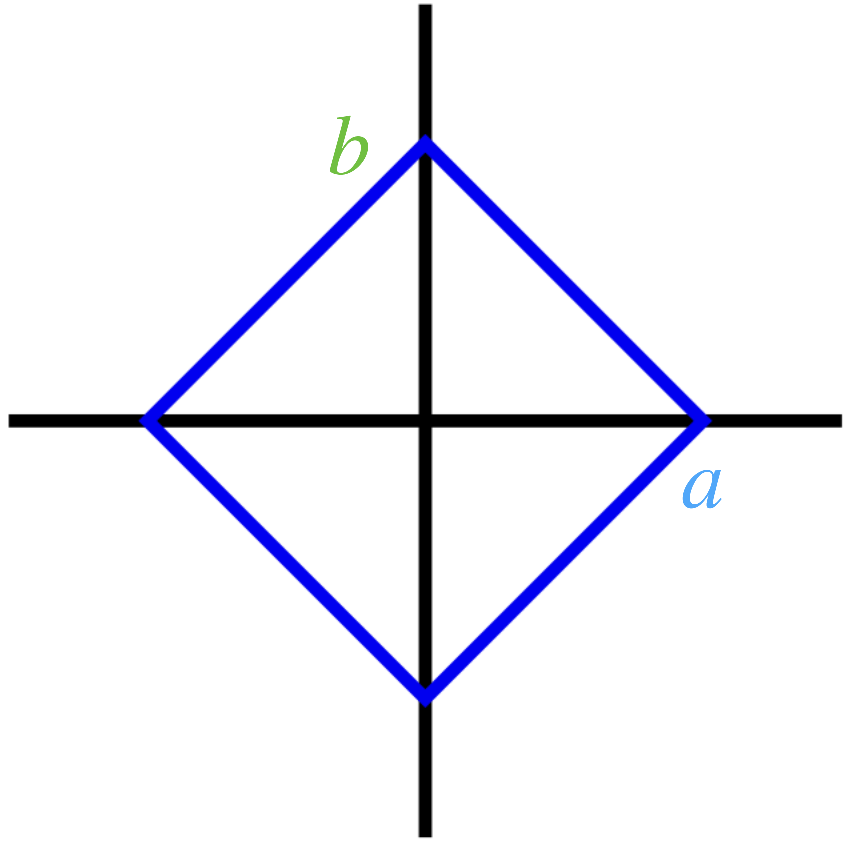

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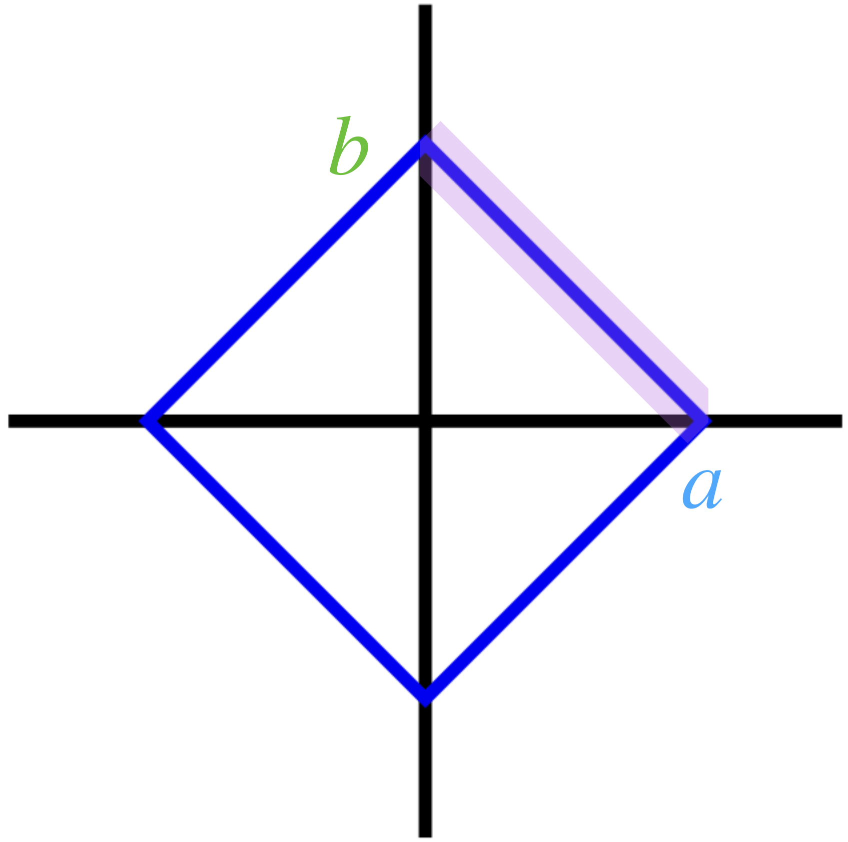

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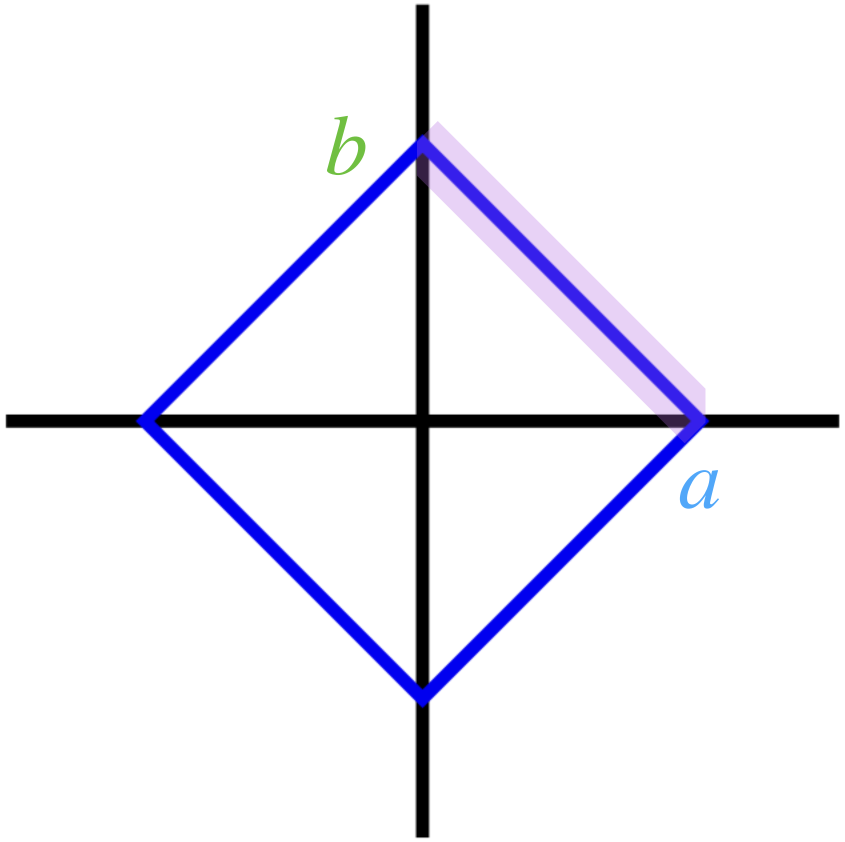

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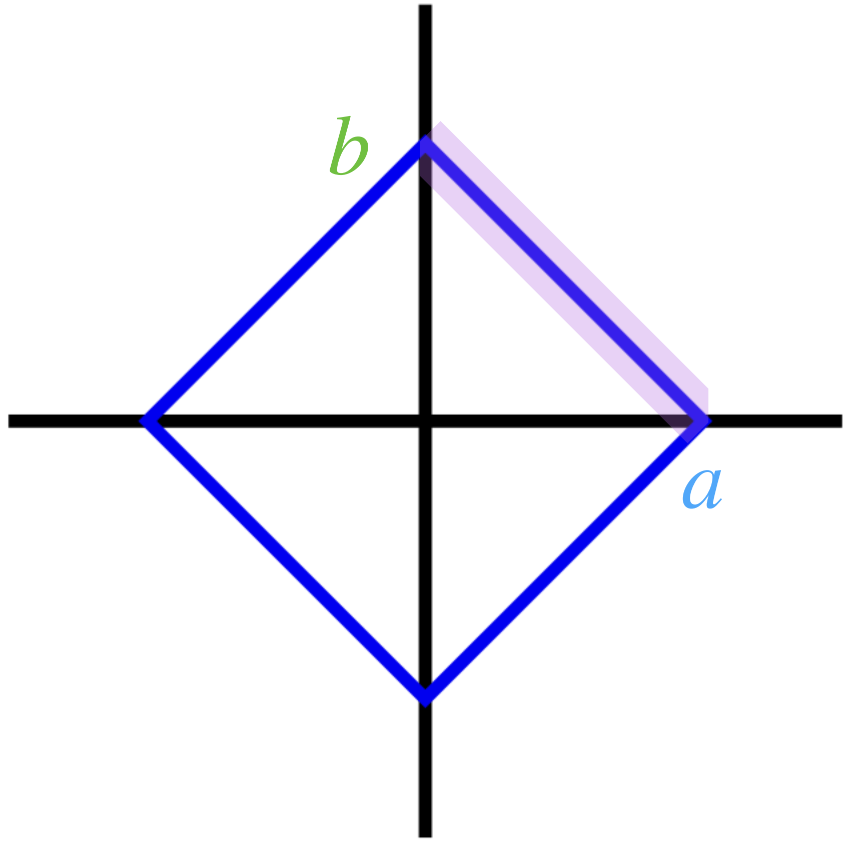

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Example: All top dimensional faces of the Whitehead link are fibered, but the vertices are not.

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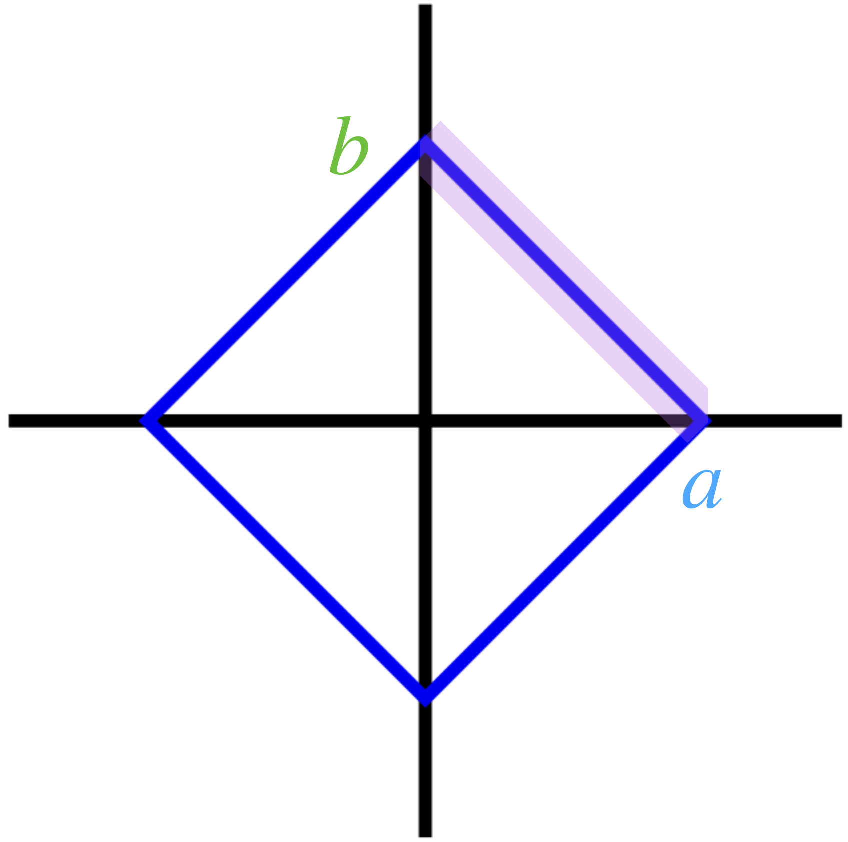

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Question: When does the Thurston norm ball of M have a fibered face?

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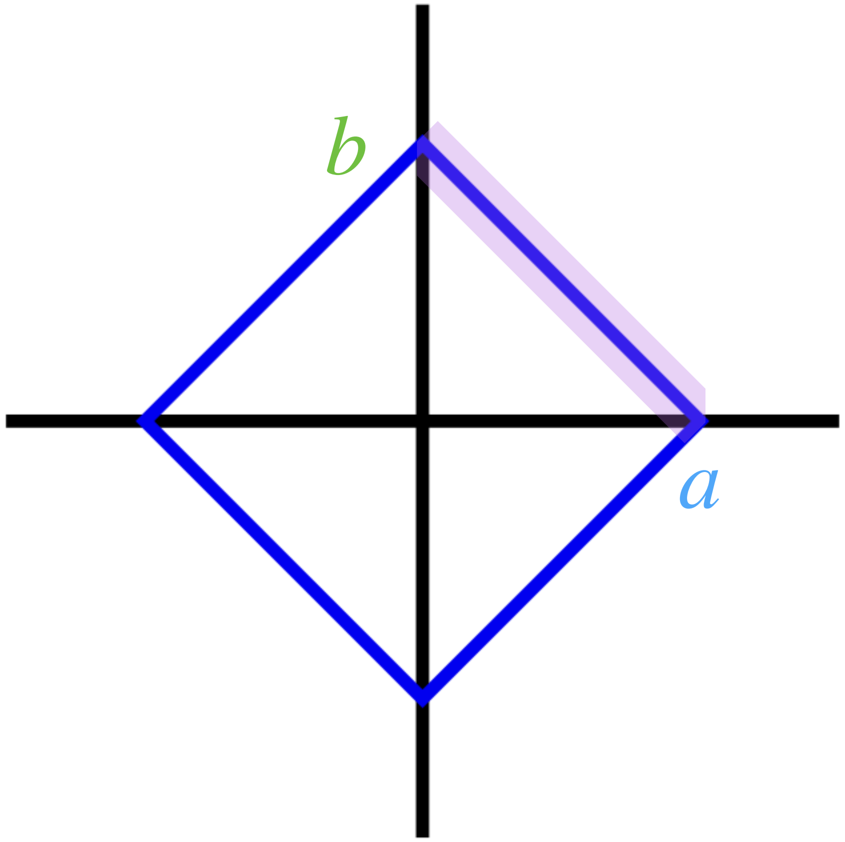

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(When is a knot fibered?)

Virtual fibering

Defn. M is virtually (blank) if some finite cover of M is (blank).

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Theorem (Agol-Wise). Every compact orientable irreducible 3-manifold with infinite fundamental group is virtually fibered.

Thank you!