Taut sutured handlebodies as twisted homology products

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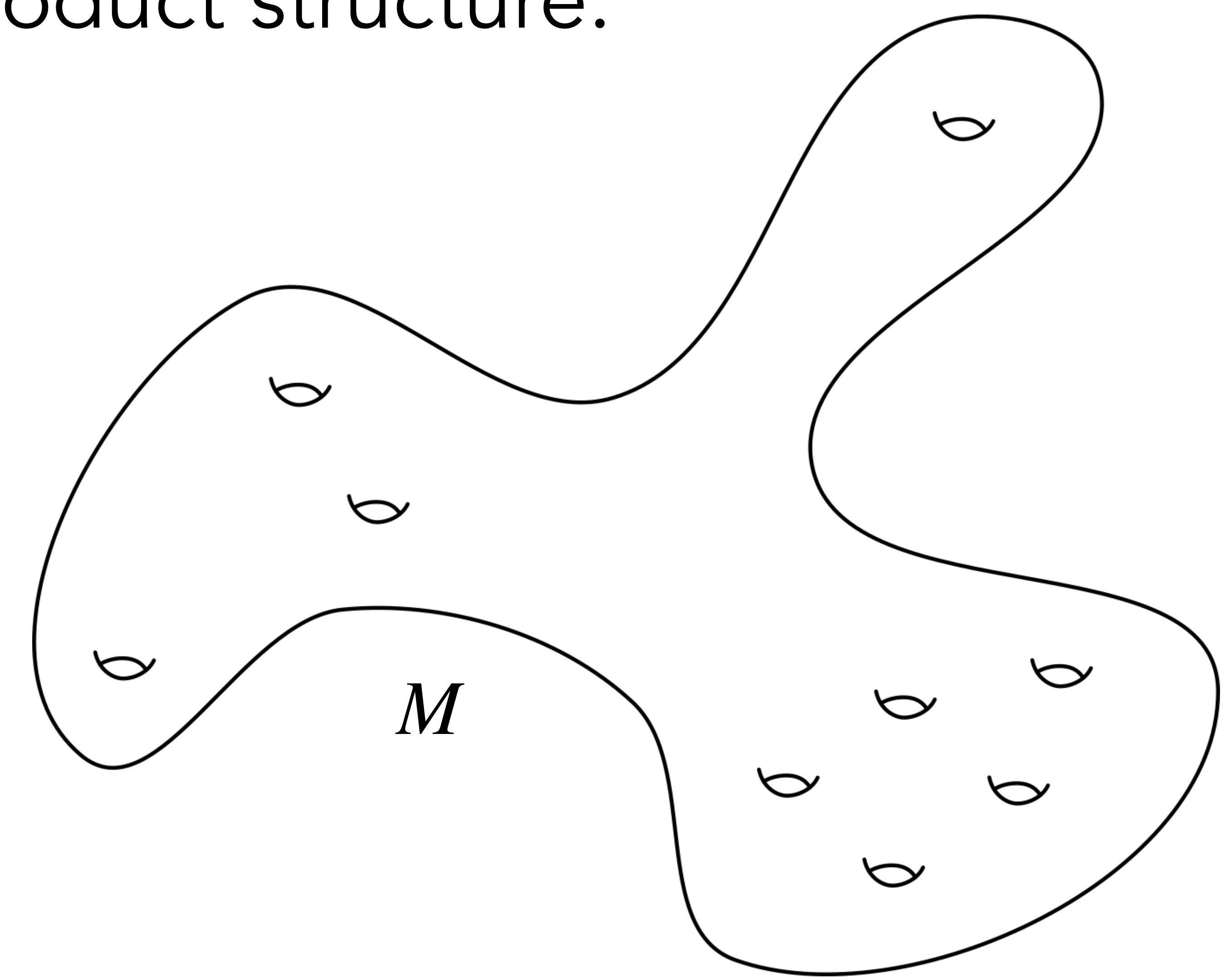
Foliations on 3-mflds

A (codimension-1) *foliation* $\mathscr{F}$ on M is a local product structure:



Foliations on 3-mflds

A foliation $\mathscr{F}$ is *taut* if there exists a loop transversely intersecting every leaf.

Question: When does M admit taut foliation?

(v2) When is $S \subset M$ a closed leaf of a taut foliation?

Foliations on 3-mflds

Question (v2):

When is $S \subset M$ a leaf of a taut foliation?

(Assume M closed or torus ∂.)

Sutured manifolds

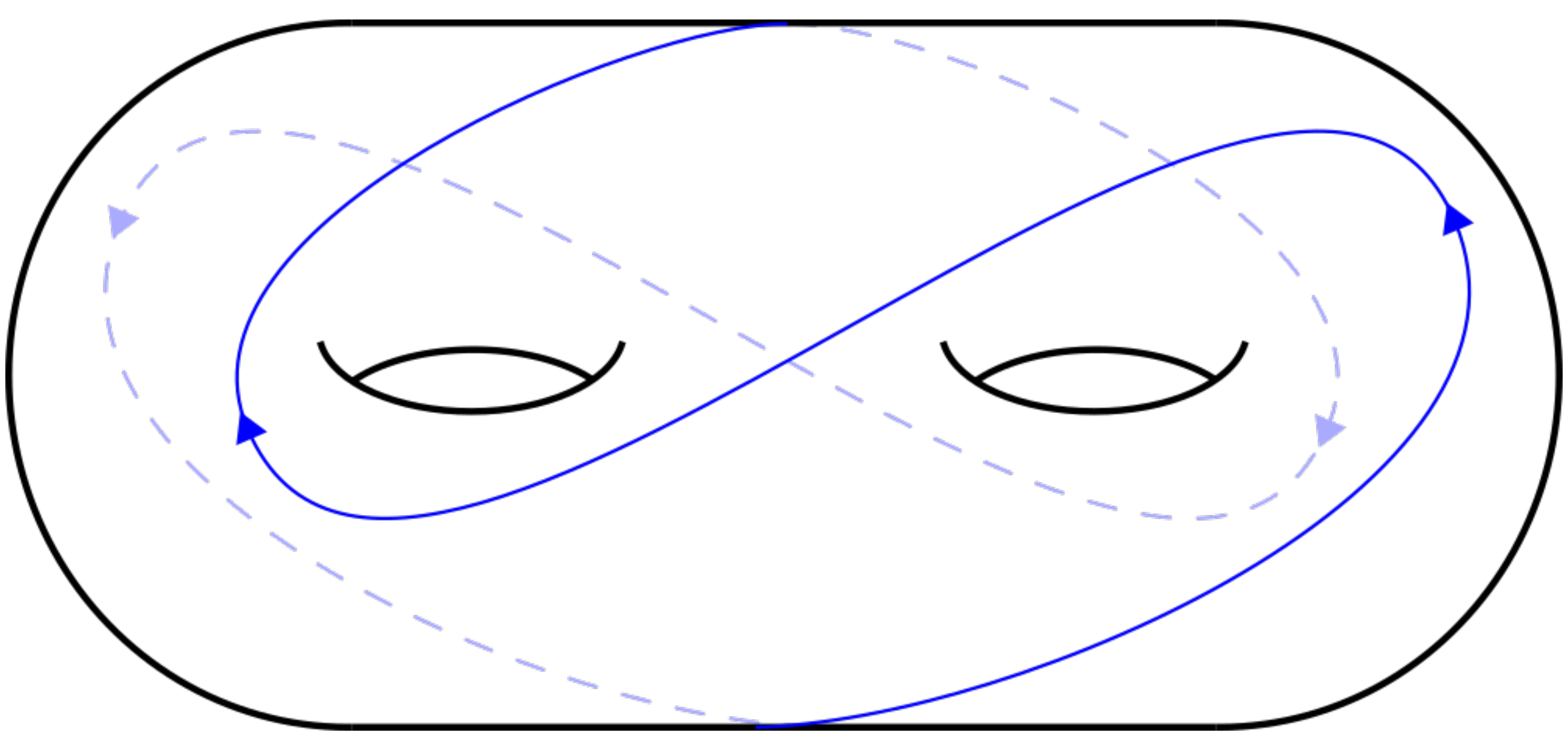
A *sutured manifold* $(M, R_\pm, \gamma)$ consists of:

M - 3-manifold (w. ∂)

γ - disjoint collection of (oriented) scc in ∂M

$R_\pm$- (oriented) subsurfaces of ∂M s.t.

$$\partial M = R_+ \sqcup_\gamma R_-$$



Sutured manifolds

Example: $M = S^3 - K$

$S \subseteq M$ Seifert surface for K

Thurston norm

Norm on $H_2(M, \partial M; \mathbb{R})$

Given $(S, \partial S) \subset (M, \partial M)$, define

$$\chi_-(S) = \max\{0, -\chi(S)\} \qquad S \text{ connected}$$

$$\chi_-(S) = \chi_-(S_1) + \chi_-(S_2) \qquad S = S_1 \sqcup S_2$$

The *Thurston norm* of $\alpha \in H_2(M, \partial M; \mathbb{Z})$ is

$$\|\alpha\| = \min_{[S]=\alpha} \chi_-(S)$$

Example: $S \subset S^3 - K$ a Seifert surface

$$\|[S]\| = 2g(K) - 1$$

Tautness

$(M, R_{\pm}, \gamma)$ is *taut* if:

- M is irreducible
- $R_{\pm}$ are incompressible in M
- $R_{\pm}$ realize Thurston norm

Fact: M admits taut $\mathscr{F}$ w. leaf S $\iff$ $M - S$ is taut

Question: How can we can we certify a sutured manifold is taut?

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Easy case

Question: How can we can we certify a sutured manifold is taut?

Obs. If M is a *product*, it is taut.

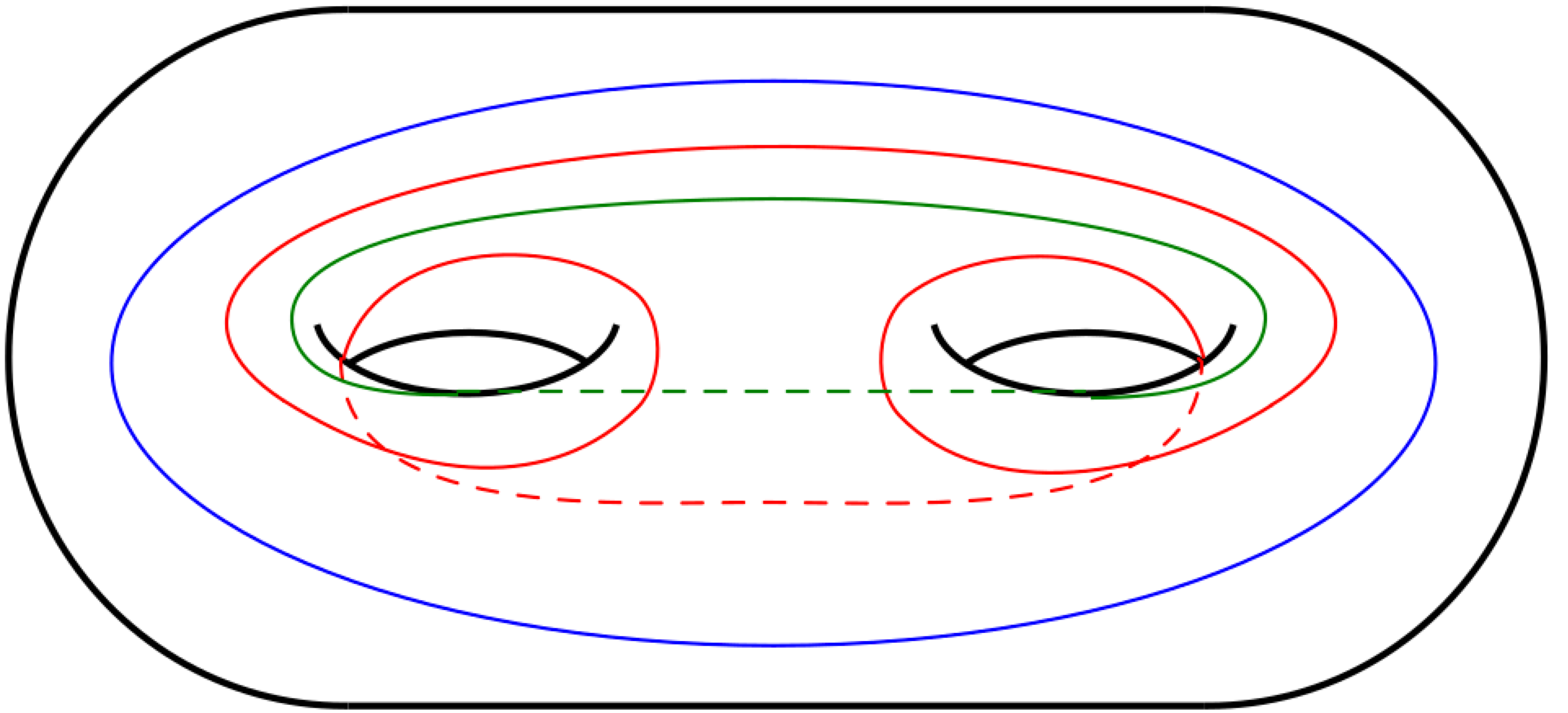
$$M \cong R_+ \times I \simeq R_+$$

Theorem (Friedl-Kim): If $i_* : H_*(R_\pm; \mathbb{Q}) \xrightarrow{\cong} H_*(M; \mathbb{Q})$, then M is taut.

M is a ($\mathbb{Q}$-)*homology product*.

A counterexample

Example (N.): M is not a $\mathbb{Q}$-homology product!



Generators of $\pi_1(R_+)$ have same image in $H_1(M)$.

Twisted homology

Core idea: Homology groups which carry additional info from a representation $\alpha : \pi_1(M) \to \mathrm{GL_n}(\mathbb{C})$.

$H_i(M; E_\alpha)$ is a $\mathbb{Z}[\pi_1(M)]$-module.

Key fact: Every theorem about (co)homology still holds here.

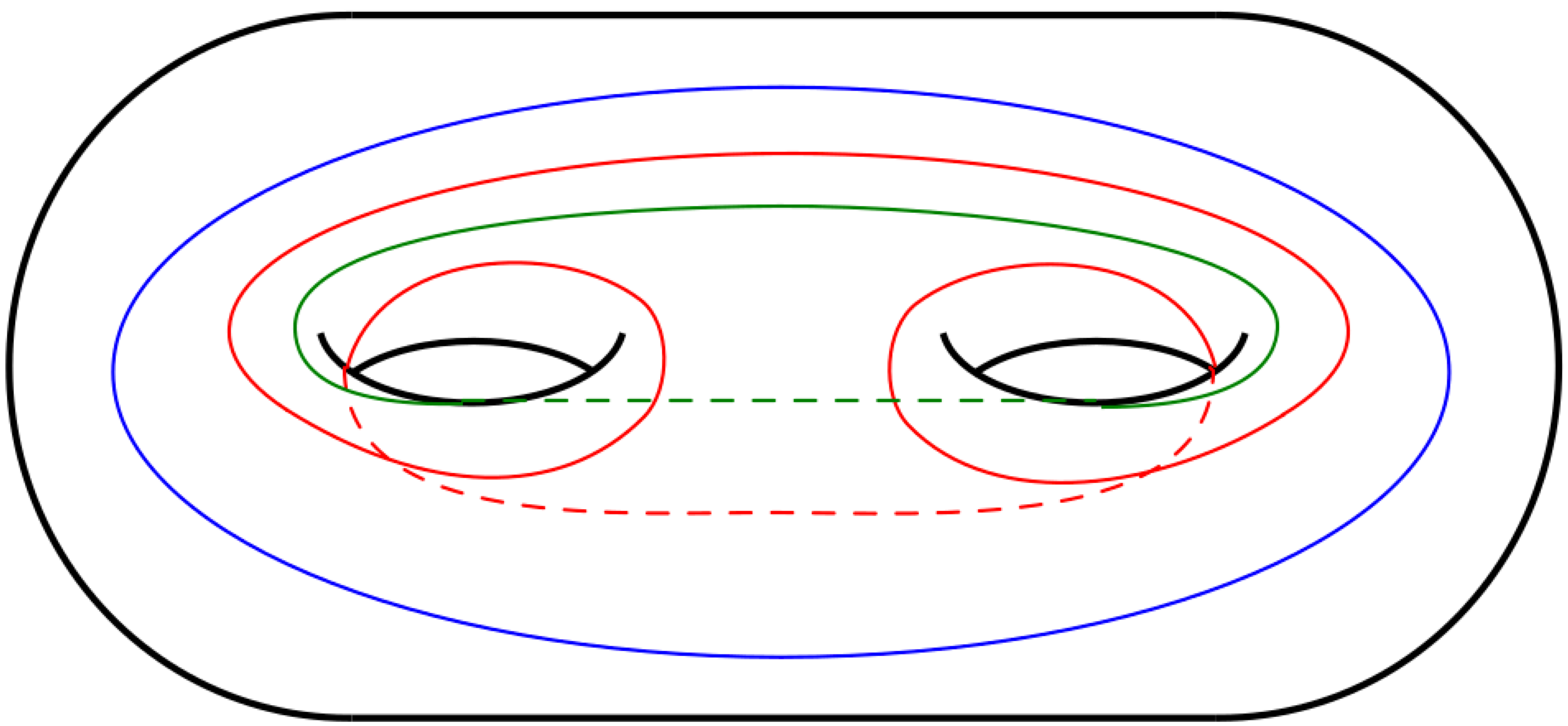
Twisted products

Theorem (Friedl-Kim): If $i_* : H_*(R_\pm; E_\alpha) \xrightarrow{\cong} H_*(M; E_\alpha)$,

then M is taut.

M is an *α-homology product*.

Example (N.): M is not a $\mathbb{Q}$-homology product…



Twisted products

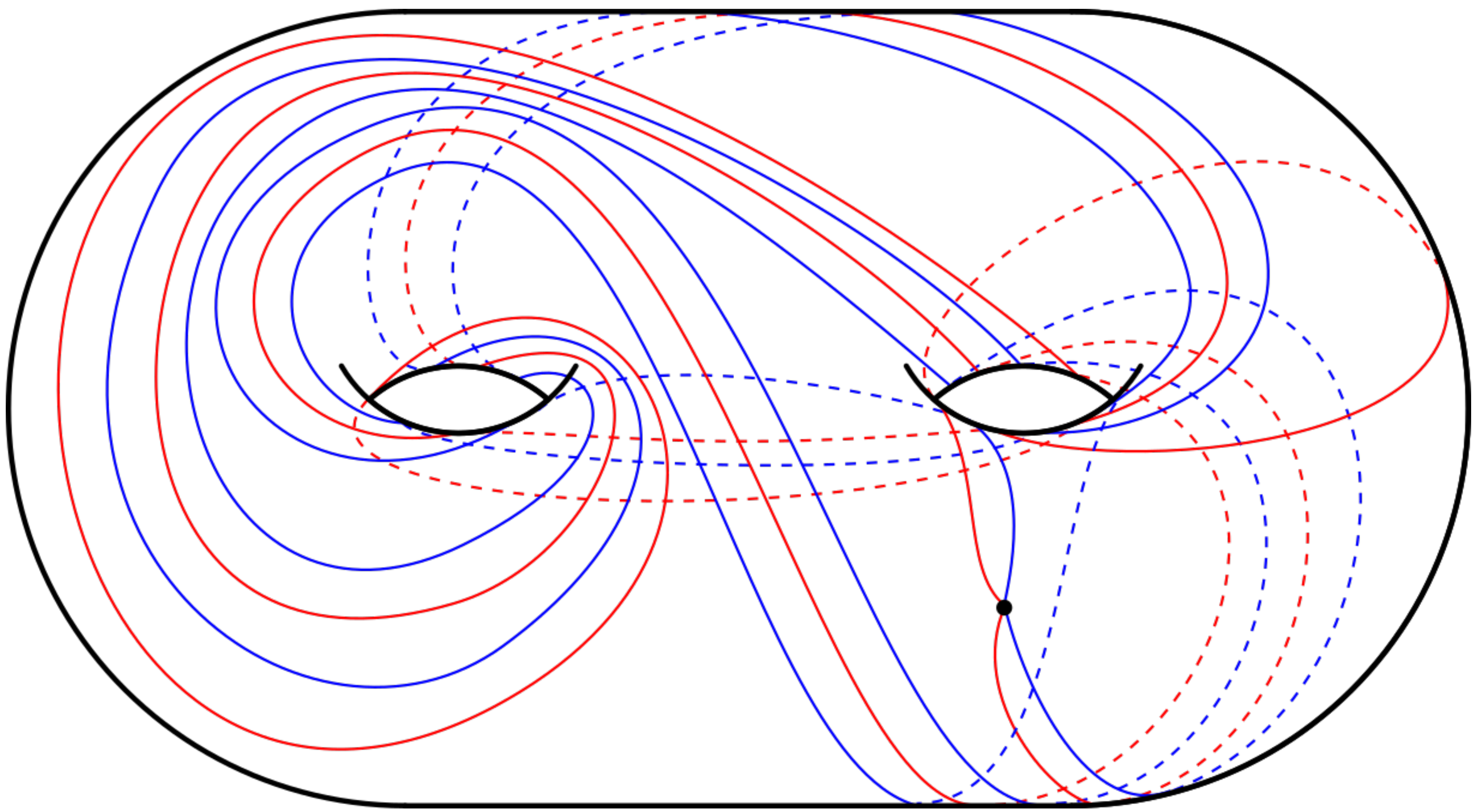
How much better is twisted homology?

Theorem (Friedl-Kim): If M is taut, there exists a rep $\alpha : \pi_1(M) \to \mathrm{GL_n}(\mathbb{C})$ such that M is an α-homology product.

The catch: The proof uses virtual fibering to produce a “good” representation.

Theorem 1

Theorem (N.): For all $g \geq 2$, there exists a taut sutured genus g handlebody M which is not an α-homology product for any $\alpha : \pi_1(M) \to \mathrm{GL}_1(\mathbb{C})$.



Main question

Can we give a precise description of the complexity of a certifying rep?

What if we restrict the algebraic properties of the rep?

Theorem 2

Theorem (N.): For all k, there exists a taut sutured handlebody M which is not an α-homology product for any representation $\alpha : \pi_1(M) \to \mathrm{GL(V)}$, with α solvable of degree $\leq k$.

Idea: Inductive construction; derived series.

Theorem 2

Corollary (N.): For all n, there exists a taut sutured handlebody M which is not an α-homology product for any solvable representation $\alpha : \pi_1(M) \to \mathrm{GL_n}(\mathbb{C})$.

Remark: For M a handlebody, Friedl-Kim's construction gives a solvable representation!