TAUT SUTURED HANDLEBODIES AS TWISTED HOMOLOGY PRODUCTS

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Abstract. Friedl and Kim show any taut sutured manifold can be realized as a twisted homology product, but their proof gives no practical description of how complicated the realizing representation needs to be. We give a number of results illustrating the relationship between the topology of a taut sutured handlebody and the complexity of a representation realizing it as a homology product.

1. Introduction

A *sutured* 3-manifold (M, γ) is a manifold with boundary marked by a set of sutures, γ, which consists of oriented curves dividing ∂M into oriented collections of components R_+ and R_-.[1]

Gabai [Gab84] introduced the notion of a *taut* sutured manifold (M, γ), which, roughly speaking, requires M to be irreducible and the boundary components $R_\pm$ to be of minimal complexity.

Under suitable hypotheses, a sutured manifold is taut if R_+ and R_- realize the Thurston norm of their (common) homology class. Here is an important example. Suppose K is a knot in S^3, and let R be a Seifert surface for K. Cutting S^3 open along R produces a sutured manifold M whose boundary decomposes along the knot K into two copies $R_\pm$ of R. This sutured manifold is taut precisely when R is of minimal genus. Thus the theory of sutured manifolds can be (and is) used to compute knot genus.

Suppose M is a sutured manifold, and $\alpha : \pi_1(M) \to \mathrm{GL}(V)$ is a representation. Then α restricts to representations $\pi_1(R_\pm) \to \mathrm{GL}(V)$, and we can define the twisted homology groups $H_*(M; E_\alpha)$, $H_*(R_\pm; E_\alpha)$. We say that M is an *α-homology product* if the maps $H_*(R_\pm; E_\alpha) \to H_*(M; E_\alpha)$ induced by inclusion are all isomorphisms. If α is not specified, we say M is a *twisted homology product*.

This concept is important, because of

Theorem 1.1 (Friedl-Kim [FK13])**.** *If M is a twisted homology product, it is taut.*

Conversely, using Agol's Virtual Fibering Theorem ([Ago08]), they show

Theorem 1.2 (Friedl-Kim [FK13])**.** *If M is taut, it is a twisted homology product for some representation α.*

We call such an α *certifying* for M. The result of Friedl and Kim is not effective, in the sense that it gives no upper or lower bounds for the complexity of a certifying representation. This potentially reduces the practical value of twisted homology as a tool. Therefore, the fundamental question we study in this paper addresses precisely this issue:

Question 1.3. *If M is a taut sutured manifold, what is the simplest representation for which it is a twisted homology product, and what is the relationship of the complexity of the representation to the topology of M?*

For M a hyperbolic manifold, Agol and Dunfield found substantial computer evidence that M is a twisted homology product for the geometric representation $\pi_1(M) \to \mathrm{SL}_2(\mathbb{C})$ ([AD15]). They conjectured in general

[1]We note Gabai's original definition allowed sutures to consist of entire torus components of the boundary. Here we are interested in sutured handlebodies, and this aspect of the definition never arises.

that every taut M has a 2-dimensional certifying representation, and proved this for a simple class of manifolds, namely books of I-bundles.

For a given M the search for a certifying representation falls into two parts: understanding the linear representations of $\pi_1(M)$, and understanding when such a representation is certifying. To simplify the discussion we restrict attention to the case that M is a handlebody, so that $\pi_1(M)$ is free.

This case is of practical importance, since it often happens that the complement of a minimal genus Seifert surface is a handlebody.

1.1. **Statement of Results.** The results herein primarily take the form of lower bounds on the complexity of a certifying representation. Our first theorem demonstrates the sharpness of the bound conjectured by Agol and Dunfield.

Theorem 1.4. *For all $g \geq 2$, there are taut sutured handlebodies M_g of genus g which fail to be a twisted homology product for any one-dimensional representation.*

Our construction for genus $g \geq 3$ exploits a condition on how $\pi_1(R_\pm)$ sit inside $\pi_1(M)$ which prevents M from being a one-dimensional twisted homology product.

The genus 2 example was found by a computer search. This example has a suture set consisting of three curves. Note that in genus two, $R_\pm$ will either be pairs of pants, or once-punctured tori. A similar search has produced no examples with $R_\pm$ once-punctured tori, which leads us to the following conjecture.

Conjecture 1.5. *Let M be a taut sutured genus-two handlebody with a single connected suture. Then M is a twisted homology product for some representation $\alpha : \pi_1(M) \to \mathrm{GL}_1(\mathbb{C})$.*

Besides the computational evidence for this conjecture, allowing only a single suture imposes a stronger relationship between $\pi_1(R_+)$ and $\pi_1(R_-)$, which we expect simplifies the situation so that, roughly speaking, less can go wrong. For example, by Mayer-Vietoris, the twisted homology groups in this setting satisfy a splitting $H_1(\partial M; E_\alpha) = H_1(R_+; E_\alpha) \oplus H_1(R_-; E_\alpha)$, which does not occur with multiple suture curves.

One might ask if, within this simplest setting, twisted coefficients are even necessary; perhaps tautness is already detected by rational homology. This is not the case, as we illustrate in Example 3.2.

We generalize the obstruction from the proof of Theorem 1.4 to obstructions for admitting solvable representations of arbitrarily large derived length. We use this to prove the following strong negation of Agol and Dunfield's conjecture within the restricted setting of solvable representations.

Theorem 1.6. *There exist taut sutured manifolds M_k such that M_k is not a twisted homology product for any solvable representation $\alpha : \pi_1(M_k) \to \mathrm{GL}_{\varphi(k)}(\mathbb{C})$, where $\varphi(k) \to \infty$ with k.*

Dropping the requirement that the representation be solvable, these examples are certified by some two-dimensional representation.

Remark 1.7. The manifolds M_k are handlebodies, which have free, and therefore residually finite rationally solvable (RFRS), fundamental group. The representations produced by Friedl and Kim in their proof of Theorem 1.2 are in general virtually solvable, and in the case the fundamental group of the sutured manifold is RFRS, solvable on the nose. This theorem demonstrates the inherent weakness in their approach, if one hopes to find tight bounds on the minimal dimension of a certifying representation.

The paper is organized as follows. In Section 2, we briefly review the theory of taut sutured manifolds and taut sutured manifolds. In Section 3, we give two examples illustrating the need for twisted coefficients.

Section 4 gives an algebraic condition for being a twisted homology product, and then addresses specifically the situation of one-dimensional representations, including conditions for being a one-dimensional twisted homology product. We use these conditions in Section 5 to prove Theorem 1.4. Finally, in Section 6 we generalize the techniquess of Sections 4 and 5 to prove Theorem 1.6.

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2. Basic definitions and facts

2.1. Sutured manifolds.

Definition 2.1. A *sutured manifold* is a four-tuple $(M, R_\pm, \gamma)$ consisting of a compact 3-manifold M and a collection of pairwise disjoint, embedded curves $\gamma \subset \partial M$, which partition $\partial M - \gamma$ into oriented subsurfaces R_+ and R_-, such that the orientations induced on their common boundary γ agree.

Though this definition does not require it, we will always assume M is connected. Some sources define the sutures to be a collection of annuli; our definition as a collection of curves is equivalent, though we occasionally view the sutures as annuli when convenient for notational or conceptual purposes.

Example 2.2.

(1) Given any compact surface S, the manifold $M = S \times I$ can be given a natural sutured structure, where $\gamma = \partial S \times I$, $R_+ = S \times 1$ and $R_- = S \times 0$.
(2) Any Seifert surface S associated to a knot K, or more generally a link L, defines a sutured manifold $S^3 - N(S)$, with $\gamma = K$ (or L) and $R_\pm \cong S$. The knot (or link) is fibered by S exactly when this sutured manifold is a product.

We are particularly interested in *taut* sutured manifolds, which we define below. We recall first the *Thurston norm* on $H_2(M, N \subseteq \partial M)$. Given a connected embedded surface $(S, \partial S) \subseteq (M, N)$, we define $\chi_-(S) = \max\{0, -\chi(S)\}$. For S not connected, $\chi_-(S) = \sum_{T \subseteq S} \chi_-(T)$, taken over connected components of S. Finally, the Thurston norm of $\sigma \in H_2(M, N)$ is defined as

$$\|\sigma\| = \min_{[S]=\sigma} \chi_-(S).$$

Definition 2.3. A sutured manifold M is *taut* if it is irreducible and $R_\pm$ are *taut*, that is, they are incompressible and realize the Thurston norm of their homology class.

Definition 2.4. A sutured manifold M is *balanced* if it is irreducible and $\chi(R_+) = \chi(R_-)$, and moreover M is not a solid torus without sutures, and if any component of $R_\pm$ has positive Euler characteristic, then M is D^3 with a single suture.

Notice that a taut sutured manifold is necessarily balanced. We will make use of this prerequisite, in particular that $\chi(R_+) = \chi(R_-)$.

2.2. **Twisted homology products.** Associated to any representation $\alpha : \pi_1(M) \to \mathrm{GL}(V)$ of the fundamental group of a sutured manifold M are homology groups $H_*(M; E_\alpha)$ and cohomology groups $H^*(M; E_\alpha)$ with coefficients twisted by the representation α. More briefly, we refer to these groups as the α-twisted (co)homology of M. The notation E_α refers to the view of the twisted coefficients as a vector bundle equipped with the action of $\pi_1(M)$ via α.

The inclusions $i_\pm : R_\pm \to M$ pullback the coefficient bundle E_α to a bundle over $R_\pm$, allowing us to define $H_*(R_\pm; i_\pm^* E_\alpha)$ along with natural maps

$$H_*(R_\pm; i_\pm^* E_\alpha) \xrightarrow{(i_\pm)_*} H_*(M; E_\alpha),$$

and similarly in cohomology. Notice any representation α restricts to representations $(i_\pm)_*\alpha : \pi_1(R_\pm) \to \mathrm{GL}(V)$, which is exactly the action defining $i_\pm^* E_\alpha$. We will generally elide the pullback notation and write $H_*(R_\pm; E_\alpha)$.

Definition 2.5. A sutured manifold M is an *α-homology product* for a representation $\alpha : \pi_1(M) \to \mathrm{GL}(V)$ if the maps $i_*^\pm$ are all isomorphisms.

This is equivalent to requiring all *relative* twisted homology groups vanish:

$$H_*(M, R_\pm; E_\alpha) = 0.$$

Our interest in twisted homology products is motivated by the following theorem of Friedl and Kim ([FK13]).

Theorem 2.6 (Friedl-Kim)**.** *Let M be a balanced sutured manifold. Then M is taut if and only if M is an α-homology product for some $\alpha : \pi_1(M) \to \mathrm{GL}_n(\mathbb{C})$.*

In particular, the representation α may always be taken to be a unitary representation. This proves any taut sutured manifold can be realized as a twisted homology product, giving a novel method for verifying tautness of sutured manifolds. However, their construction of the certifying representation uses in a key way Agol's virtual fibering ([Ago08]).

Most standard homological tools translate to the setting of twisted coefficients. with one noteworthy caveat accompanying the twisted version of the universal coefficient theorem. The identifications here are

$$H^k(M; E_\alpha) \cong H_k(M; E_\alpha^*) \quad \text{and} \quad H^k(M, R; E_\alpha) \cong H_k(M, R; E_\alpha^*),$$

where E_α^* is the *dual bundle* to E_α. This bundle corresponds to the 'dual representation' α^* defined as the unique representation such that $\langle \alpha(g^{-1})v, w\rangle = \langle v, \alpha^*(g)w\rangle$ for all $v, w \in \mathrm{GL}(V)$ and $g \in \pi_1(M)$.

In general, the bundles E_α and E_α^* will not be isomorphic. We will often be interested in representations which satisfy the following homological generalization of this condition.

Definition 2.7. A representation $\alpha : \pi_1(M) \to \mathrm{GL}(V)$ is *homologically self-dual* if, for any subspace $A \subseteq M$, there is an isomorphism $H_*(M, A; E_\alpha) \cong H^*(M, A; E_\alpha)$.

For example, any unitary representation is homologically self-dual, as is any representation to $SL_2(K)$, for any field K. This condition is of particular use because it greatly simplifies verifying M as a twisted homology product.

Proposition 2.8 (Agol-Dunfield [AD15], Proposition 3.1)**.** *Suppose M is a connected, balanced sutured manifold with $R_\pm$ nonempty. If α is homologically self-dual, then M is an α-homology product if and only*

if any one of the following vanish:

$$H_k(M, R_\pm; E_\alpha),\ H^k(M, R_\pm; E_\alpha) \text{ for } k = 1, 2.$$

We give their proof to highlight a couple of facts which do not need the assumption of homological self-duality.

Proof. As $R_\pm$ are nonempty, we know $H_0(M, R_\pm; E_\alpha) = H^0(M, R_\pm; E_\alpha) = 0$. By Poincaré duality, also $H_3(M, R_\mp; E_\alpha) = 0$. Now suppose $H_1(M, R_-; E_\alpha) = 0$; the other cases are similar. Since M is balanced, we have $\chi(R_\pm) = \chi(M)$, so $\chi(H_*(M, R_-; E_\alpha)) = 0$. Then, since $H_k(M, R_-; E_\alpha) = 0$ for $k \neq 2$, we also have $H_2(M, R_-; E_\alpha)) = 0$. Poincaré duality now shows $H^*(M, R_+; E_\alpha) = 0$. Finally, as α is homologically self-dual, this gives $H_*(M, R_+; E_\alpha) = H^*(M, R_+; E_\alpha) = 0$. □

We do not use self-duality until the last step. More generally, we can say

Corollary 2.9. *For $R = R_\pm$,*

$$H_1(M, R; E_\alpha) = 0 \quad \Longleftrightarrow \quad H_2(M, R; E_\alpha) = 0,$$

and

$$H^1(M, R; E_\alpha) = 0 \quad \Longleftrightarrow \quad H^2(M, R; E_\alpha) = 0.$$

Corollary 2.10. *M is an α-homology product if and only if*

$$H_1(M, R_+; E_\alpha) = H_1(M, R_+; E_\alpha^*) = 0$$

for either choice of $R = R_\pm$.

In particular, if M is an α-homology product, it is also an α^*-homology product.

2.3. **Sutured manifold hierarchies.** To conclude this section, we discuss one method we might try to use for constructing representations, and why it fails. Recall the sutured manifold hierachy of a taut sutured manifold M is a sequence of decompositions

$$M = M_0 \xrightarrow{S_1} M_1 \xrightarrow{S_2} M_2 \xrightarrow{S_3} \cdots \xrightarrow{S_n} M_n$$

such that each S_k meets the sutures of M_{k-1} transversally, each M_k is taut, and every embedded surface in M_n is separating. Gabai introduced this concept in [Gab83], proving such hierarchies always exist, and moreover, that if a sequence of decompositions of an arbitrary sutured manifold M satisfies certain additional conditions, tautness of M_n implies M is taut as well.

As these hierarchies are often used in inductive arguments, one might hope that such a hierarchy can be used to inductively construct certifying representations. More precisely, if $M \xrightarrow{S} N$ is a decomposition, then N is a subspace of M, so a representation of M restricts to a representation of N. Suppose M and N are both taut, and that α is certifying for M. One might naïvely imagine that the restriction of α is certifying for N. This is not true, as the following example shows.

Example 2.11. The handlebodies M and N in Figure 1 are related by a decomposition along a disk meeting the sutures in M in four points. In this case, we may realize $\pi_1(M)$ as an HNN extension of $\pi_1(N) \cong F_2$, with $\pi_1(M) \cong F_3$ gaining a free generator z. The representation $\alpha : \pi_1(M) \to \mathrm{GL}(\mathbb{C})$ defined by $\alpha(x) = -1$ and $\alpha(y) = \alpha(z) = 1$ is certifying for M, as can be verified via Proposition 4.1. However, when restricted to N, the representation α is no longer certifying: the locus of representations which fail to be certifying are those with $x \mapsto -1$.

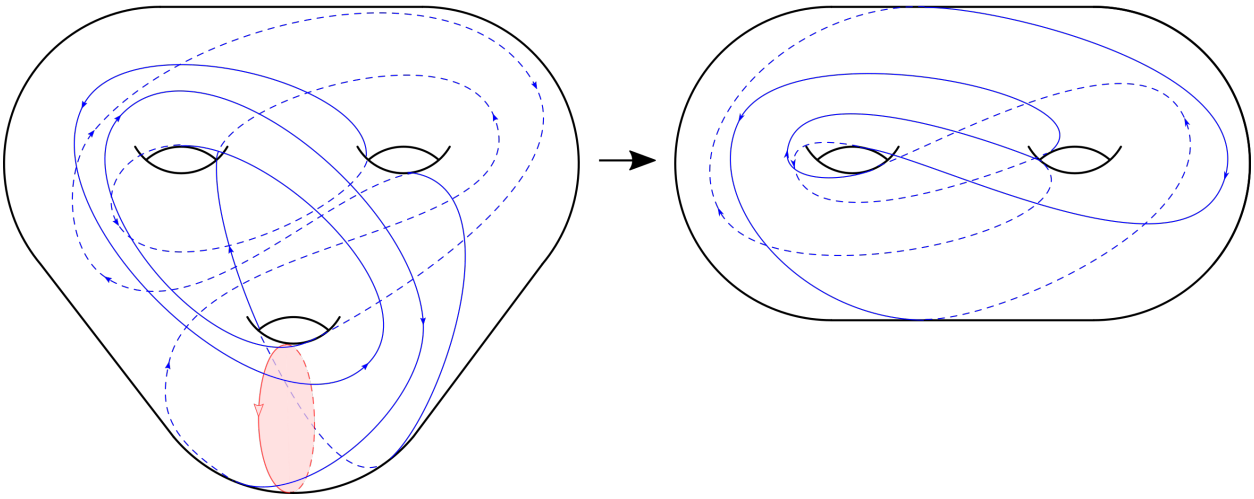


FIGURE 1. The decomposition of M (left) along the disk S to obtain N (right).

The reason for this is that there is part of the boundary of N which is not contained in the boundary of M. Understanding when this naïve guess fails requires analyzing how the suture structure changes with this new boundary, which is subtle in practice. However, this failure is isolated to the local situation of the decomposition. That is, if $S_\pm \subseteq N$ are the two copies of S in the boundary of N, there is still an injection $H_*(R_\pm - S_\pm; E_{\alpha|_N}) \hookrightarrow H_*(N; E_{\alpha|_N})$.

In this example, it is the case that both manifolds admit one-dimensional certifying representations. However, even the condition for admitting a one-dimensional certifying representation is subtle to understand in relation to a decomposition $M \xrightarrow{S} N$.

As we will see in Lemma 5.1, in the special case that the surface S is a disk meeting the sutures of M exactly twice, a certifying representation for N can be extended to one which certifies M.

3. Necessity of twisted coefficients

We give two examples of genus-two taut sutured handlebodies which fail to be rational homology products. This illustrates the necessity of twisted coefficients for certifying tautness, even in this topologically simple setting. Our first example captures the essential feature of this failure, that significant information may be lost in abelianizing an injection $\pi_1(R_\pm) \to \pi_1(M)$ to the induced map on homology $H_1(R_\pm; \mathbb{Q}) \to H_1(M; \mathbb{Q})$.

Example 3.1. Let M be a genus-two handlebody, with suture γ consisting of the three curves shown in Figure 2. These correspond to the free homotopy classes yx, $xaby$, and $(xaby^2x)^{-1}$.

The boundary components $R_\pm$ are topological pants. Their fundamental groups, as subgroups of $\pi_1(\partial M)$, are both freely generated by yx and $xaby$. These inject into $\pi_1(M)$ as the subgroup $\langle xy, yx \rangle$. Abelianizing, we see this is not a rational homology product: the generators of the fundamental group map to the same cycle in $H_1(M; \mathbb{Q})$.

The suture set in the above example consists of three curves. We can also produce examples with only a single suture curve, though none as simple as the example above.

Example 3.2. Consider M as in Figure 3. The generators of R_+ map to x and $[x, y][x, y^{-1}]$ in $\pi_1(M)$. Under the map on homology $H_1(R_+; \mathbb{Q}) \to H_1(M; \mathbb{Q})$ induced by the inclusion $i : R_+ \hookrightarrow M$, the second generator is killed. Thus its image has rank one, and so M cannot be a rational homology product.

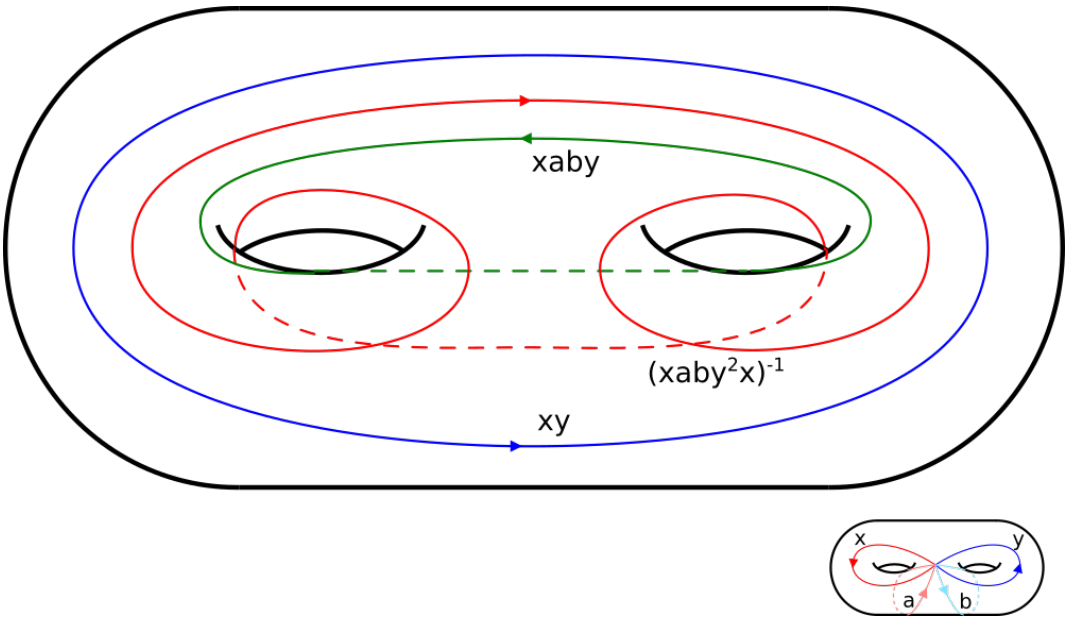


FIGURE 2. A simple example of a genus-2 taut handlebody.

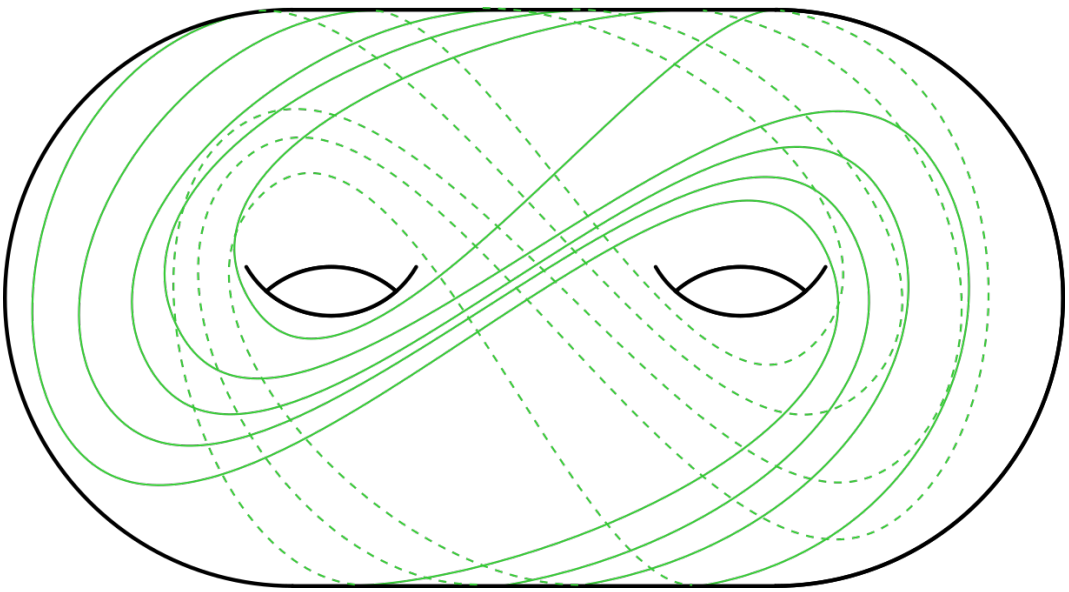


FIGURE 3. A genus-2 example with a single suture. The suture curve γ has image $[x,[x,y][x,y^{-1}]] \in \pi_1(M)$, with $\pi_1(M)$ generated by x, y as in Figure 2.

We return to these examples in Section 4 to prove tautness using tools developed therein. Alternatively, they can both be seen to be taut by observing that if not, each suture component would need to bound a disk in M, but in both examples, the suture sets contain a disk-busting curve.

Remark 3.3. These examples can be extended to any higher genus by attaching *sutured one-handles* (see Section 5 for a definition). By Lemma 5.1, this is still taut, and still fails to be a rational homology product.

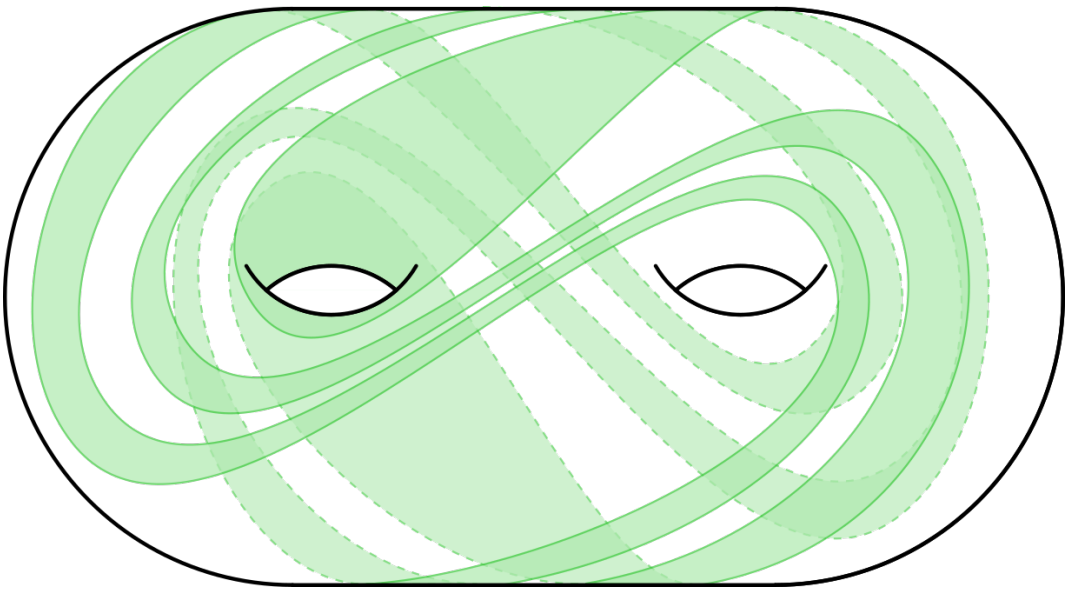


FIGURE 4. The subsurface R_+.

4. Restricting to one-dimensional representations

The following Proposition is a straightforward generalization of Proposition 5.2 of [AD15], in which $g = n = 2$. Our proof follows analogously to that of Agol-Dunfield. Take M to be a balanced sutured handlebody of genus g, with R_+ connected. Then $\pi_1(M)$ and $\pi_1(R_+)$ are free groups of rank g.

Let $\pi_1(M) = \langle x_1, \dots, x_g \rangle$ and $\pi_1(R_+) = \langle a_1, \dots, a_g \rangle$, and let $i_* : \pi_1(R_+) \to \pi_1(M)$ be the map induced by the inclusion $i : R_+ \hookrightarrow M$. Given a word $w \in \pi_1(M)$, we write $\partial_{x_i} w$ for its Fox derivatives in $\mathbb{Z}[x_1, \dots, x_g]$ ([Fox53]). Notice that any representation $\alpha : \pi_1 M \to \mathrm{GL}(V)$ extends naturally to a ring homomorphism $\alpha : \mathbb{Z}[x_1, \dots, x_g] \to \mathrm{End}(V)$.

Proposition 4.1. *For a fixed representation $\alpha : \pi_1(M) \to \mathrm{GL}(V)$, with $\dim V = n$, if the sutured handlebody M is an α-homology product, then the $gn \times gn$ matrix*

$$\Big(\alpha\big(\partial_{x_i} i_*(a_j)\big)\Big)_{i,j}$$

has nonzero determinant.

Furthermore, when α is homologically self-dual, this condition is sufficient.

Proof. Let W be a two-complex with a single vertex v, $2g$ edges $e_{x_1}, \dots, e_{x_g}$ and $e_{a_1}, \dots, e_{a_g}$, and g faces $r_1, \dots, r_g$, which are attached to the edges according to $i_*(a_i)a_i^{-1}$, for each i. Set $B = \bigcup_i e_{a_i}$. Then there is a map $j : (W, B) \to (M, R_+)$ realizing the map $\pi_1(W) \to \pi_1(M)$ sending $[e_{x_i}] \mapsto x_i$ and $[e_{a_i}] \mapsto a_i$, and which is a homotopy equivalence of these pairs of spaces.

The map $j_* : \pi_1(W) \to \pi_1(M)$ pulls α back to a representation $\alpha \circ j_* : \pi_1(W) \to \mathrm{GL}(V)$. The map j then induces maps between the twisted homology and cohomology groups associated to α:

$$H_*(M; E_\alpha) \to H_*(W; E_{\alpha \circ j_*}), \qquad H^*(W; E_{\alpha \circ j_*}) \to H^*(M; E_\alpha),$$
$$H_*(R_+; E_\alpha) \to H_*(B; E_{\alpha \circ j_*}), \qquad H^*(B; E_{\alpha \circ j_*}) \to H^*(R_+; E_\alpha).$$

As j is a homotopy equivalence $M \to W$ and $R_+ \to B$, these maps are all isomorphisms. The isomorphisms on cohomology combine with the long exact sequence for pairs to give the following commutative diagram.

$$\begin{tikzcd}
H^{n-1}(W; E_{\alpha\circ j_*}) \arrow[r, "j_*"] \arrow[d, "i_*"'] & H^{n-1}(M; E_\alpha) \arrow[d, "i_*"'] \\
H^{n-1}(B; E_{\alpha\circ j_*}) \arrow[r, "j_*"] \arrow[d, "d"'] & H^{n-1}(R_+; E_\alpha) \arrow[d, "d"'] \\
H^n(W, B; E_{\alpha\circ j_*}) \arrow[r, dashed] \arrow[d] & H^n(M, R_+; E_\alpha) \arrow[d] \\
H^n(W; E_{\alpha\circ j_*}) \arrow[r, "j_*"] \arrow[d, "i_*"'] & H^n(M; E_\alpha) \arrow[d, "i_*"'] \\
H^n(B; E_{\alpha\circ j_*}) \arrow[r, "j_*"] & H^n(R_+; E_\alpha)
\end{tikzcd}$$

By the five lemma, the induced map $j_* : H_*(M, R_+; E_\alpha) \to H_*(W, B; E_{\alpha\circ j_*})$ is also an isomorphism.

If M is an α-homology product, then $H_2(M, R_-; E_\alpha) \cong H^1(M, R_+; E_\alpha) = 0$, and also $H^1(W, B; E_{\alpha\circ j_*}) = 0$. To define $H^1(W, B; E_{\alpha\circ j_*})$, we begin with the chain complex $C_*(\widetilde{W}; \mathbb{Z})$ of $\mathbb{Z}[\pi_1(M)]$-modules associated to the universal cover of W, then take $\mathbb{Z}[\pi_1(M)]$-module homomorphisms of this complex to V. Writing $\Lambda = \mathbb{Z}[\langle x_1, \ldots, x_g\rangle] = \mathbb{Z}[\pi_1(M)]$, the chain complex has the form

$$C_*(\widetilde{W}; \mathbb{Z}) : \quad 0 \to \bigoplus_i \Lambda r_{a_i} \xrightarrow{\partial_2} \bigoplus_i (\Lambda e_{x_i} \oplus \Lambda e_{a_i}) \xrightarrow{\partial_1} \Lambda v \to 0.$$

The left-module map ∂_i can be represented as a matrix, which act on an element of $C_i(\widetilde{W}; \mathbb{Z})$, viewed as a row vector, by multiplication on their left and the vector's right, namely $\partial_i(u) = u \cdot \partial_i$. These matrices are

$$\partial_1 = \begin{pmatrix} x_1 - 1 \\ \vdots \\ x_g - 1 \\ i_*(a_1) - 1 \\ \vdots \\ i_*(a_g) - 1 \end{pmatrix} \qquad \partial_2 = \begin{pmatrix} \partial_{x_1} i_*(a_1) & \cdots & \partial_{x_g} i_*(a_1) & -1 & & \\ \vdots & \ddots & \vdots & & \ddots & \\ \partial_{x_1} i_*(a_g) & \cdots & \partial_{x_g} i_*(a_g) & & & -1 \end{pmatrix}$$

Note the left half of the second matrix consists of the Fox derivatives $\partial_{x_i} i_*(a_j) a_j^{-1} = \partial_{x_i} i_*(a_j)$; similarly, the entries in the right half are $\partial_{a_i} i_*(a_j) a_j^{-1} = -\delta_{i,j}$.

Applying $\mathrm{Hom}_\Lambda(\cdot, V)$ to $C_*(\widetilde{W}; \mathbb{Z})$ gives the cochain complex $C^*(W; E_{\alpha\circ j_*})$. The effect of applying this functor replaces each Λ with a copy of V, and applying α (extended to a ring homomorphism) to each element of the matrices representing ∂_1 and ∂_2 to obtain the d^i maps. As matrices, the d^i act by multiplication on column vectors to their right. Then the cochain complex is

$$C^*(W; E_{\alpha\circ j_*}) : \quad 0 \leftarrow V^g \xleftarrow{d^1} V^{2g} \xleftarrow{d^0} V \leftarrow 0.$$

We are interested, however, in $C^*(W, B; E_{\alpha\circ j_*})$. This consists of those cochains which vanish when restricted B, namely, which are supported away from B. As B is one-dimensional, this consists of all of $C^2(W; E_{\alpha\circ j_*})$, as well as the cochains supported on the e_{x_i} in $C^1(W; E_{\alpha\circ j_*})$. Thus the relative cochain complex is

$$C^*(W, B; E_{\alpha\circ j_*}) : \quad 0 \leftarrow V^g \xleftarrow{d^1} V^g \leftarrow 0 \leftarrow 0.$$

Here d^1 restricts to the $\alpha(\partial_{x_i} i_*(a_j))$ half of the full matrix.

As M is taut, $H^1(W, B; E_{\alpha\circ j_*}) = 0$, so d^1 must have full rank, that is, the determinant in the statement of the proposition must be nonvanishing.

Lastly, if α is homologically self-dual and this determinant is nonzero, by Corollary 2.10, M is taut. □

The condition of this determinant being nonzero corresponds exactly to $H^2(M, R_+; E_\alpha)$ (and therefore $H_1(M, R_-; E_\alpha)$) vanishing. In the case α is not homologically self-dual, we can still verify tautness by checking that neither this determinant nor that associated to α^* vanishes.

Remark 4.2. From the perspective of the representation variety $\mathrm{Hom}(\pi_1(M), \mathrm{GL}(V))$, the condition given by Proposition 4.1 determines a Zariski-open subspace of certifying representations. In the setting of M a handlebody, the representation variety is connected, and so such a subspace is either empty, or the complement of a collection of lower dimensional subvarieties, and therefore dense in the full variety.

On a practicable level, this is good news for certifying tautness. Supposing we knew an upper bound on minimal complexity of a certifying representation, we expect a 'random' representation of that complexity to in fact be certifying.

Proposition 4.1 specifically applies to sutured handlebodies, and does not immediately generalize outside of this setting. However, we expect this intuition for the space of certifying representations within the representation variety to generalize, and the certifying representations to similarly form a Zariski-open subspace. However, for M not a handlebody, this representation variety may not be connected.

Example 4.3 (Examples 3.1 and 3.2). We can apply Proposition 4.1 to see the manifolds in our earlier examples are taut. First, for (M, γ) in Example 3.1, let $\alpha : \pi_1(M) \to \mathrm{GL}_1(\mathbb{C})$ be any one-dimensional representation. By Proposition 4.1, M is an α-homology product when

$$\det \begin{pmatrix} \alpha\big(\partial_x(xy)\big) & \alpha\big(\partial_y(xy)\big) \\ \alpha\big(\partial_x(yx)\big) & \alpha\big(\partial_y(yx)\big) \end{pmatrix} \neq 0.$$

That is to say,

$$\det \begin{pmatrix} \alpha\big(\partial_x(xy)\big) & \alpha\big(\partial_y(xy)\big) \\ \alpha\big(\partial_x(yx)\big) & \alpha\big(\partial_y(yx)\big) \end{pmatrix} = \det \begin{pmatrix} \alpha(1) & \alpha(x) \\ \alpha(y) & \alpha(1) \end{pmatrix} = 1 - \alpha(xy) \neq 0.$$

Rephrasing what we saw in Example 3.1, this shows in particular we cannot take α to be the trivial representation, where $\alpha(x) = \alpha(y) = 1$. However, for any choice of α with $\alpha(xy) \neq 1$, this will not be 0.

We turn to (M, γ) from Example 3.2. For $\alpha : \pi_1(M) \to \mathrm{GL}_1(\mathbb{C})$ fixed, M is an α-homology product when

$$\det \begin{pmatrix} \alpha\big(\partial_x(x)\big) & \alpha\big(\partial_y(x)\big) \\ \alpha\big(\partial_x([x,y][x,y^{-1}])\big) & \alpha\big(\partial_y([x,y][x,y^{-1}])\big) \end{pmatrix} \neq 0.$$

These Fox derivatives are

$$\partial_x(x) = 1, \qquad \partial_y(x) = 0,$$
$$\partial_x([x,y][x,y^{-1}]) = 1 - xyx^{-1} + [x,y] - [x,y]xy^{-1}x^{-1},$$
$$\partial_y([x,y][x,y^{-1}]) = x - [x,y] - [x,y]xy^{-1} + [x,y]xy^{-1}x^{-1}.$$

The image of α is abelian, so

$$\begin{aligned}
&\det\begin{pmatrix} \alpha\big(\partial_x(x)\big) & \alpha\big(\partial_y(x)\big) \\ \alpha\big(\partial_x([x,y][x,y^{-1}])\big) & \alpha\big(\partial_y([x,y][x,y^{-1}])\big)\end{pmatrix} \\
&= \det\begin{pmatrix} 1 & 0 \\ \alpha(2-y-y^{-1}) & \alpha(x-1-xy^{-1}+y^{-1})\end{pmatrix} \\
&= \alpha\big((y^{-1}-1)(1-x)\big) \neq 0.
\end{aligned}$$

This condition is non-vanishing – specifically, whenever $\alpha(x)$, $\alpha(y) \neq 1$ – yielding a nonempty Zariski-open set of certifying α.

Returning to the setting of a genus-g sutured handlebody M, consider the case of a one-dimensional representation $\alpha : \pi_1(M) \to \mathrm{GL}_1(\mathbb{C})$. Here, we have an algebraic understanding of what it means to be a twisted homology product. For a word $w \in \pi_1(M)$, we write ∂w for the vector of Fox derivatives of w with respect to $x_1, \ldots, x_g$.

Proposition 4.4. *M is a one-dimensional twisted homology product if and only if the vectors of abelianized Fox derivatives* $\mathrm{ab}(\partial i_*(a_j))$ *are linearly independent.*

Proof. Consider the composition of maps

$$\pi_1(M) \xrightarrow{\partial} \mathbb{Z}[\pi_1(M)]^g \xrightarrow{\alpha} (\mathrm{GL}_1(\mathbb{C}))^g,$$

which takes a $a \in \pi_1(M)$ to the vector of the α-images of its g partial Fox derivatives $\partial_{x_i} a$. Since $\mathrm{GL}_1(\mathbb{C})$ is abelian, α factors through the abelianization

$$\pi_1(M) \xrightarrow{\partial} \mathbb{Z}[\pi_1(M)]^g \xrightarrow{\mathrm{ab}} \mathbb{Z}[\mathbb{Z}^g]^g \xrightarrow{\alpha} (\mathrm{GL}_1(\mathbb{C}))^g.$$

Similarly, the composition of maps

$$\pi_1(M)^g \xrightarrow{\partial} \mathbb{Z}[\pi_1(M)]^{g^2} \xrightarrow{\alpha} (\mathrm{GL}_1(\mathbb{C}))^{g^2} \xrightarrow{\det} \mathbb{C},$$

factors

$$\pi_1(M)^g \xrightarrow{\partial} \mathbb{Z}[\pi_1(M)]^{g^2} \xrightarrow{\mathrm{ab}} \mathbb{Z}[\mathbb{Z}^g]^{g^2} \xrightarrow{\alpha} (\mathrm{GL}_1(\mathbb{C}))^{g^2} \xrightarrow{\det} \mathbb{C}$$

We claim for any $w \in \mathbb{Z}[\mathbb{Z}^g] = \mathbb{Z}[\pi_1(M)^{\mathrm{ab}}]$, we can choose α to detect w, meaning $\alpha(w) \neq 0$. Order the monomial terms of w by setting $\prod x_i^{n_i} > \prod x_i^{m_i}$ if $n_i > m_i$ for the first i such that $n_i \neq m_i$. Then pick α so that $\alpha(x_1) \gg \alpha(x_2) \gg \cdots \gg \alpha(x_g)$; the leading term of w will dominate, and $\alpha(w) \neq 0$.

Because α is one-dimensional, the following diagram commutes.

$$\begin{tikzcd}
\mathbb{Z}[\mathbb{Z}^g]^{g^2} \arrow[r, "\alpha"] \arrow[d, "\det"] & (\mathrm{GL}_1(\mathbb{C}))^{g^2} \arrow[d, "\det"] \\
\mathbb{Z}[\mathbb{Z}^g] \arrow[r, "\alpha"] & \mathrm{GL}_1(\mathbb{C})
\end{tikzcd}$$

Thus $\det(\mathrm{ab}(\partial i_*(a_j))) = 0$ exactly when $\det(\alpha(\partial i_*(a_j))) = 0$ for all choices of α. □

We end this section with a lemma which provides a condition for being a one-dimensional twisted product. It will prove useful for finding non-examples. Recall the *derived series* $G^{(k)}$ of G is defined by $G^{(0)} = G$ and $G^{(k+1)} = [G^{(k)}, G^{(k)}]$.

Lemma 4.5. *If M is a one-dimensional twisted homology product, then $\pi_1(R_\pm) \cap \pi_1(M)^{(2)} \subseteq \pi_1(R_\pm)^{(1)}$.*

Proof. Suppose M is an α-homology product for some $\alpha : \pi_1(M) \to \mathrm{GL}_1(\mathbb{C})$.

Recall that $H^1(M; E_\alpha)$ is the group of all twisted homomorphisms $f : \pi_1(M) \to \mathbb{C}$, modulo twisted homomorphisms of the form $\hat{z}(g) = \alpha(g) \cdot z - z$ for $z \in \mathbb{C}$. Any $f \in H^1(M; E_\alpha)$ necessarily vanishes on $\pi_1(M)^{(2)}$. We see this first by observing that

$$\begin{aligned} f([u,v]) &= f(u) + \alpha(u)f(v) + \alpha(uv)f(u^{-1}) - \alpha(uvu^{-1})f(v^{-1}) \\ &= f(u) + \alpha(u)f(v) - \alpha(uvu^{-1})f(u) - \alpha(uvu^{-1}v^{-1})f(v) \\ &= (1-\alpha(v))f(u) - (1-\alpha(u))f(v), \end{aligned}$$

since $\mathrm{GL}_1(\mathbb{C})$ is abelian. Now, this is zero when $\alpha(u) = \alpha(v) = 1$, for instance, for $u, v \in \pi_1(M)^{(1)}$. Such elements $[u,v]$ normally generate $\pi_1(M)^{(2)}$, so f must vanish on all of $\pi_1(M)^{(2)}$.

Consider now $H^1(R_\pm; E_\alpha)$. Any twisted homomorphism is determined by its values on the generators $a_1, \ldots, a_g$ of $\pi_1(R_\pm)$. Fix $w \in \pi_1(R_\pm) \cap \pi_1(M)^{(2)}$ and let $\#_{a_i} w$ denote the number of occurrences of a_i (counted with sign) in w. Notice $\#_{a_i} w = 0$ for all i is exactly the condition for $w \in \pi_1(R_\pm)^{(1)}$. Supposing $w \notin \pi_1(R_\pm)^{(1)}$, then some $\#_{a_i} w \neq 0$. Define $g \in H^1(R_\pm; E_\alpha)$ by $g(a_j) = \delta_{ij}$. By construction, $g(w) \neq 0$.

Consider the long exact sequence of cohomology groups

$$\cdots \to H^1(M; E_\alpha) \xrightarrow{i_*} H^1(R_\pm; E_\alpha) \xrightarrow{\delta} H^2(M, R_\pm; E_\alpha) \to \cdots .$$

As any $f \in H^1(M; E_\alpha)$ vanishes on w, the twisted homomorphism g constructed above does not lie in the image of i_*. But by exactness, g then is not in the kernel of δ, so $H^2(M, R_\pm; E_\alpha) \neq 0$. By Poincaré duality, then $H_1(M, R_\mp; E_\alpha) \neq 0$, which contradicts our assumption that M is an α-homology product. □

5. Examples which are not one-dimensional homology products

In this section, we give a family of handlebodies of all genus $g \geq 2$ which are not twisted homology products for any one-dimensional representation. We begin with a lemma which describes a way of increasing genus of a taut handlebody while preserving the set of certifying representations. This is used in conjunction with a genus-two example to prove the main result of the section. The genus-two example was found via computer search with SnapPy ([CDGW]). We follow this with an explicit construction an example of a genus-three handlebody, which better elucidates the obstruction to admitting a one-dimensional certifying representation.

For ease of notation, we treat γ as a collection of annuli instead of curves. Given a sutured manifold M, we can construct a new sutured manifold N by attaching a *sutured one-handle*. The one-handle $D^2 \times D^1$ is given a product sutured structure $I \times (D^1 \times D^1)$. It is attached to M along the disks $I \times (D^1 \times \partial D^1)$, which we require to meet γ in two strips so that $0 \times (D^1 \times \partial D^1) \subset R_-$ and $1 \times (D^1 \times \partial D^1) \subset R_+$. This construction is illustrated in Figure 5.

Lemma 5.1. *Suppose M is a taut sutured manifold. If $(N, R'_\pm, \gamma')$ is obtained by attaching a sutured one-handle to M, then N is also taut. Moreover, for any representation $\alpha : \pi_1(M) \to \mathrm{GL}(V)$, there is a representation $\alpha' : \pi_1(N) \to \mathrm{GL}(V)$ with $\alpha'|_{\pi_1(M)} = \alpha$ such that M is an α-homology product if and only if N is an α'-homology product.*

Proof. Note $\pi_1(N) = \pi_1(M) * \langle x \rangle$, where x is the core of the one-handle. Moreover, $\pi_1(R'_\pm) = \pi_1(R_\pm) * \langle x \rangle$. Define α' to agree with α on $\pi_1(M)$ and to map x to the identity.

Notice

$$\det\left(\alpha'\big(\partial_{x'_i} i_*(a'_j)\big)\right) = \det\begin{pmatrix} \alpha\big(\partial_{x_i} i_*(a_j)\big) & 0 \\ 0 & I \end{pmatrix} = \det\left(\alpha\big(\partial_{x_i} i_*(a_j)\big)\right).$$

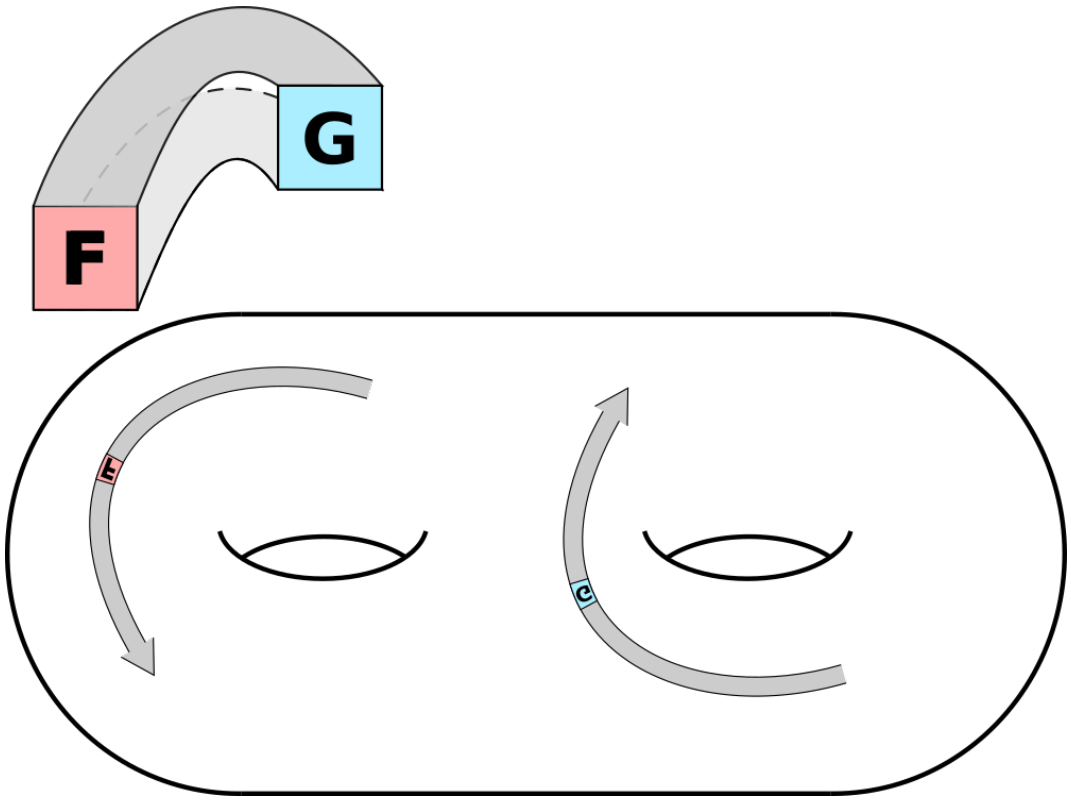


FIGURE 5. Attaching a sutured handle. The grey bands represent portions of the suture γ, viewed as annuli and oriented by the arrows drawn. Homeomorphisms F and G identify the corresponding disks. The letters F and G indicate the maps' respective orientations and restrictions to the boundary.

The corresponding equality also holds for the dual representations. Thus the result follows from Proposition 4.1. □

Theorem 5.2 (Theorem 1.4)**.** *For every $g \geq 2$, there is a taut sutured handlebody M_g of genus g such that M is not an α-homology product for any representation $\alpha : \pi_1(M_g) \to \mathrm{GL}_1(\mathbb{C})$.*

Proof. For $g = 2$, we construct $M = M_2$ as follows. Define a sutured structure on M by taking $R_+ \cong \Sigma_{0,3} \subset \partial M$ to be a tubular neighborhood of the curves illustrated in Figure 6; γ the boundary of this neighborhood; and $R_- = \partial M - R_+$. Fix x and y as generators of $\pi_1(M)$. The two boundary curves in the figure, which generate $\pi_1(R_+)$, map to $a = [y, x^{-1}][x, y^{-1}]$ and $b = [y^{-1}, x][y^{-1}, x^{-1}]$.

Let $\alpha : \pi_1(M) \to \mathrm{GL}_1(\mathbb{C})$ be any representation. Since α has abelian image, we can simplify the matrix A in Proposition 4.1 by replacing the entries with the abelianization of the Fox derivatives. This yields

$$A = \begin{pmatrix} \alpha(-x^{-1}y + x^{-1} + 1 - y^{-1}) & \alpha(1 - x^{-1} - xy^{-1} + y^{-1}) \\ \alpha(y^{-1} - 1 - x^{-1}y^{-1} + x^{-1}) & \alpha(-2y^{-1} + xy^{-1} + x^{-1}y^{-1}) \end{pmatrix}.$$

We leave it to the reader to verify the determinant of this matrix vanishes, independent of the choice of α.

We demonstrate that M is taut via the certifying representation $\beta : \pi_1(M) \to \mathrm{SL}_2(\mathbb{C})$, defined by

$$\beta(x) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad \beta(y) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

As remarked earlier, representations to $\mathrm{SL}_2(\mathbb{C})$ are homologically self-dual, so Proposition 4.1 applies.

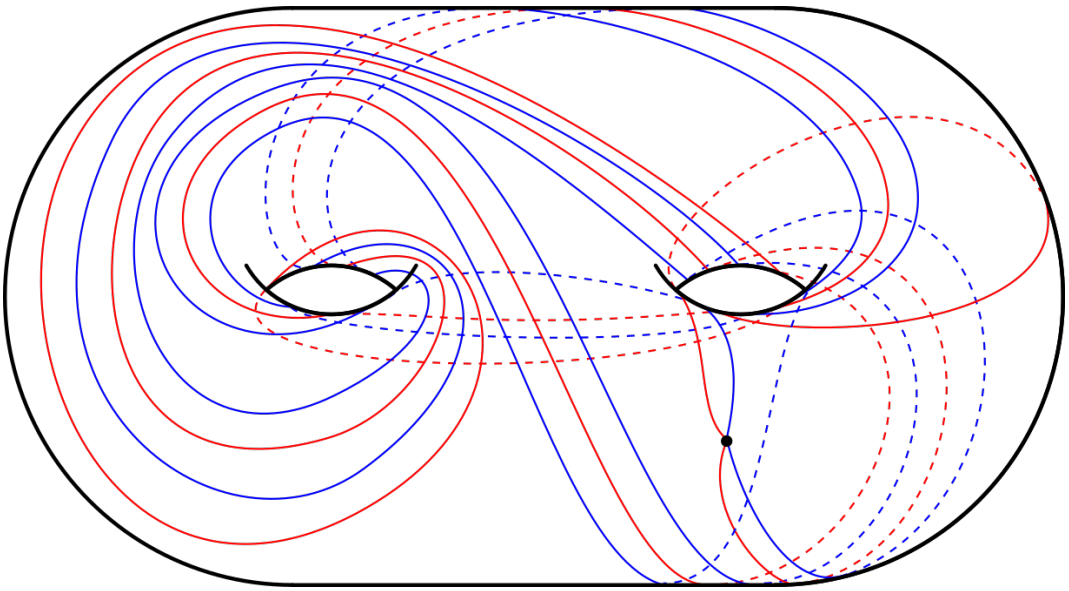


FIGURE 6. Example with no certifying 1-dimensional representation.

The (non-abelianized) Fox derivatives of a and b are

$$\begin{aligned}
\partial_x a &= -yx^{-1} + yx^{-1}y^{-1} + yx^{-1}y^{-1}x - yx^{-1}y^{-1}x^2y^{-1}x^{-1} \\
\partial_y a &= 1 - yx^{-1}y^{-1} - yx^{-1}y^{-1}x^2y^{-1} + yx^{-1}y^{-1}x^2y^{-1}x^{-1} \\
\partial_x b &= y^{-1} - y^{-1}xyx^{-1} - y^{-1}xyx^{-1}y^{-1}x^{-1} + y^{-1}xyx^{-1}y^{-1}x^{-1}y \\
\partial_y b &= -y^{-1} + y^{-1}x - y^{-1}xyx^{-1}y^{-1} + y^{-1}xyx^{-1}y^{-1}x^{-1}
\end{aligned}$$

Then

$$\det \begin{pmatrix} \beta(\partial_x a) & \beta(\partial_y a) \\ \beta(\partial_x b) & \beta(\partial_y b) \end{pmatrix} = \det \begin{pmatrix} 0 & 3 & 0 & -2 \\ 0 & 6 & -1 & -3 \\ 0 & -1 & 0 & 1 \\ 1 & 1 & 0 & -1 \end{pmatrix} = 1,$$

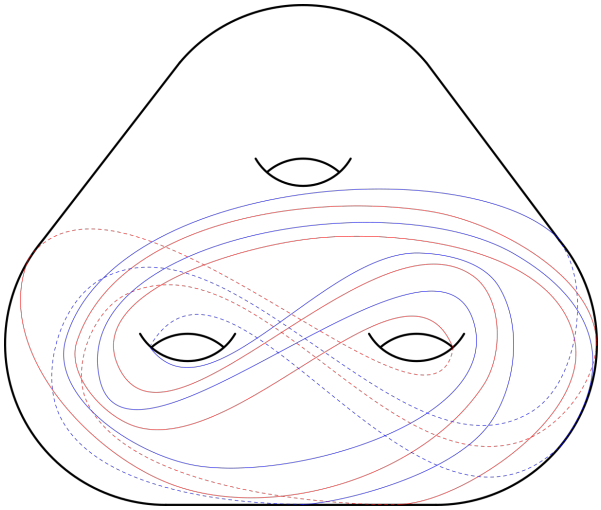
so M is taut.

We may iteratively apply Lemma 5.1 to construct higher genus handlebodies from this example. The process in the Lemma gives a handlebody M_g for all $g > 2$ which is still a two-dimensional twisted homology product, and fails to admit a certifying one-dimensional representation. □

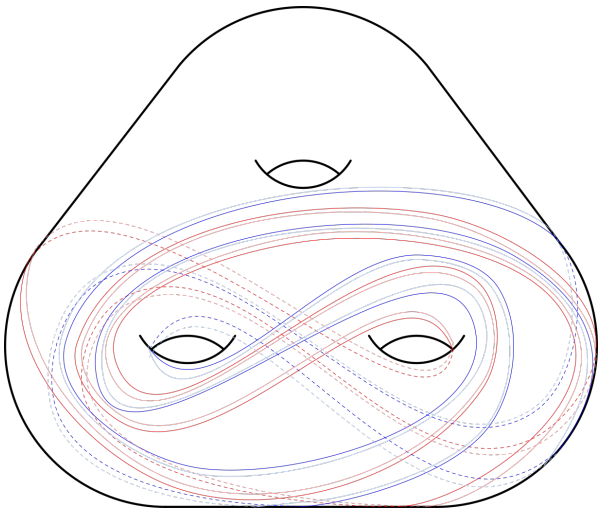
We now give an alternative, explicit construction of a genus-three example. This example puts to use Lemma 4.5, by building a curve which lies in $\pi_1(M)^{(2)}$. It is also a precursor to the construction within the proof of Theorem 6.1 in Section 6.

Example 5.3. We build a taut $(M, R_\pm, \gamma)$ with R_+ containing a curve whose image in $\pi_1(M)$ lies in $\pi_1(M)^{(2)}$. Let M be a genus-3 handlebody, with $\pi_1(M) = \langle x, y, z \rangle$.

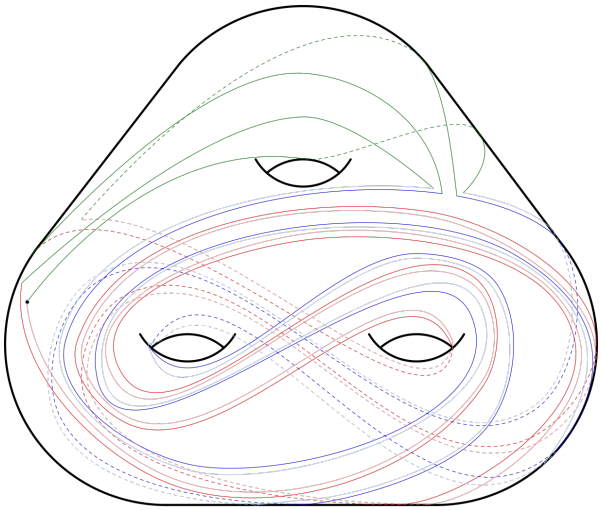
To describe the sutured structure on M, we begin by constructing a simple closed curve a on the boundary of M which lives in $\pi_1(M)^{(2)}$. Figure 7 illustrates this process. First, we draw the curves A and B, which are disjoint and have image in $\pi_1(M)^{(1)}$. The curve a is constructed from A and B to have image $a = [A, zBz^{-1}] \in \pi_1(M)$. Figures 7b and 7c show this construction, by first taking two copies of each A and B, and then connecting them via arcs to yield a simple closed curve with the desired image. Picking a basepoint



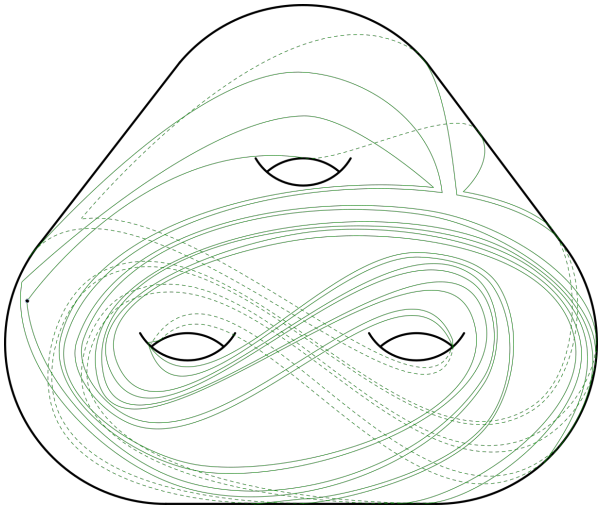
(A) Disjoint curves A, B with image in $\pi_1(M)^{(1)}$.



(B) A, B doubled and pushed off themselves.



(C) Copies of A and B connected by segments z.



(D) The curve $a = [A, zBz^{-1}]$.

FIGURE 7. Construction of the curve a.

along a, we then find two more simple closed curves b and c on ∂M, disjoint away from the basepoint, as shown in Figure 8. This captures all the information we need to define $(M, R_\pm, \gamma)$: a neighborhood of this defines R_+, which is homeomorphic to $\Sigma_{1,2}$, its boundary γ, and its complement R_-. From the construction, we see

$$\mathrm{Im}(\pi_1(R_+)) = \langle \left[[x,y][x^{-1},y], z[y^{-1},x][y,x]z^{-1}\right], [x,y][y^{-1},x^{-1}], z\rangle.$$

We now check our example is taut. We do this by exhibiting a two-dimensional representation $\beta : \pi_1(M) \to \mathrm{GL}_2(\mathbb{C})$ which realizes M as a twisted homology product. Define β as follows:

$$\beta(x) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad \beta(y) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \beta(z) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

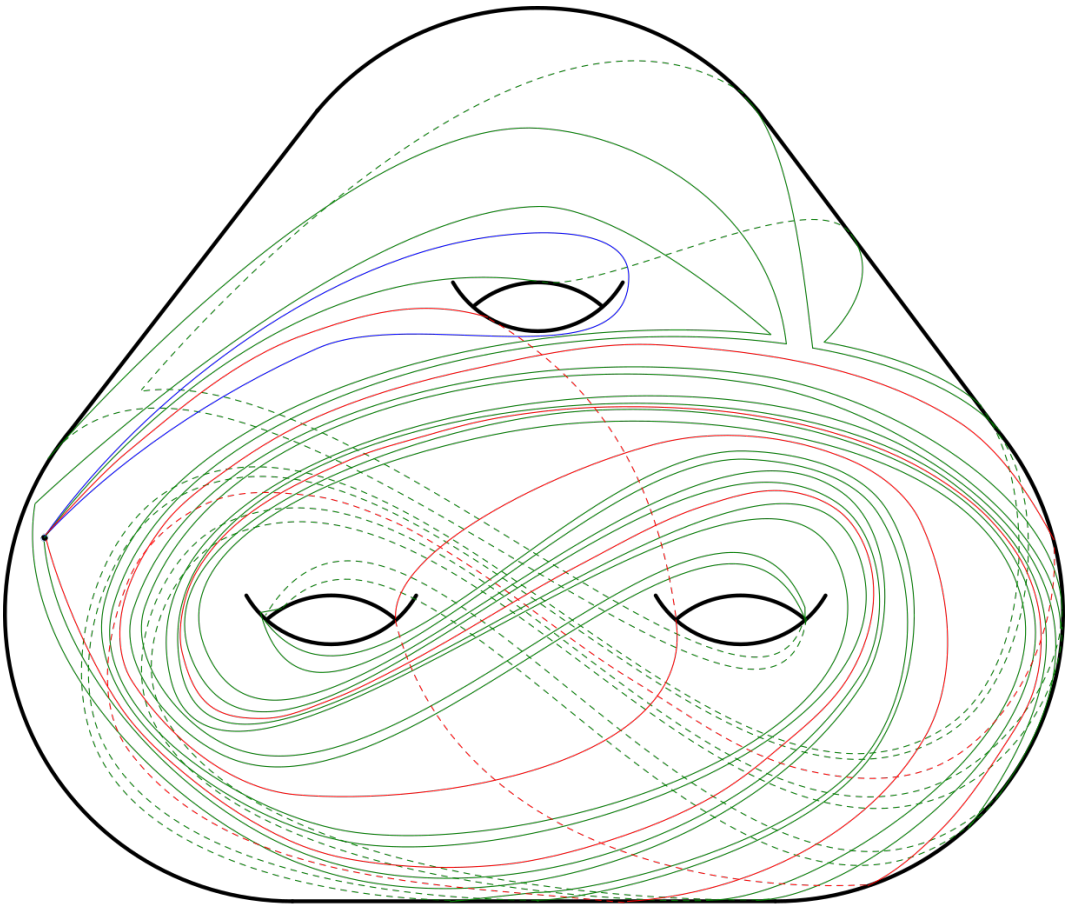


FIGURE 8. Curves a, b, and c.

In fact β is a representation to $U(2)$, so the associated twisted homology is self-dual. Then we can apply Proposition 4.1. The relevant matrix is

$$\begin{pmatrix} \beta\big(\partial_x i_*(a)\big) & \beta\big(\partial_y i_*(a)\big) & \beta\big(\partial_z i_*(a)\big) \\ \beta\big(\partial_x i_*(b)\big) & \beta\big(\partial_y i_*(b)\big) & \beta\big(\partial_z i_*(b)\big) \\ \beta\big(\partial_x i_*(c)\big) & \beta\big(\partial_y i_*(c)\big) & \beta\big(\partial_z i_*(c)\big) \end{pmatrix} = \begin{pmatrix} -8 & -36 & 7 & 23 & 37 & 4 \\ -2 & -7 & 1 & 5 & 8 & 1 \\ 4 & 2 & -1 & -1 & 0 & 0 \\ 2 & 3 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

which is invertible with determinant 32, so M is indeed taut.

6. Restricting to solvable representations

In this section, we restrict to the setting of solvable representations. A group G is *solvable* if its derived series $G^{(k)} = [G^{(k-1)}, G^{(k-1)}]$ has finite length. For G solvable, let K denote the length of this series, that is, the smallest index k such that $G^{(k)} = 1$. Then G is solvable of *degree* K. This is equivalent to realizing G as a K-fold abelian extension of an abelian group. We say a representation $\alpha : G \to \mathrm{GL}(V)$ is *solvable* if it has solvable image, and similarly define the *degree of solvability* of α to be the degree of solvability of its image.

In this section, we prove the following.

Theorem 6.1. *For any K, there is a taut sutured handlebody M_K which fails to be a twisted homology product for any solvable representation of degree less than K.*

Observe that Example 3.1 and Theorem 5.2 satisfy this Theorem for $K = 1, 2$, respectively. In the setting of $\mathrm{GL}_n(\mathbb{C})$, Zassenhaus shows for a fixed n any solvable subgroup is of bounded degree of solvability ([Zas37]). Let $\varphi(K)$ denote the smallest n for which $\mathrm{GL}_n(\mathbb{C})$ admits a solvable subgroup of degree K.

Corollary 6.2 (Theorem 1.6)**.** *The handlebody M_K is not a twisted homology product for any solvable representation to $\mathrm{GL}_n(\mathbb{C})$ for $n < \varphi(K)$.*

In particular, the conjecture of Agol and Dunfield is false when restricted to the class of solvable representations.

The next lemma captures the connection between solvability of a representation and its behavior with respect to the Fox derivative.

Lemma 6.3. *If $\alpha : G \to \mathrm{GL}(V)$ is solvable of degree K, then $\alpha(\partial g) = 0$ for any $g \in G^{(K+1)}$.*

Proof. We show this holds for $g = [g_1, g_2]$ where $g_1, g_2 \in G^{(K)}$; as elements of this form generate $G^{(K+1)}$, this suffices. Recall

$$\partial g = \partial g_1 + g_1 \partial g_2 - g_1 g_2 g_1^{-1} \partial g_1 - g_1 g_2 g_1^{-1} g_2^{-1} \partial g_2.$$

As $\alpha(g_1) = \alpha(g_2) = 1$, thus

$$\alpha(\partial g) = \alpha(\partial g_1) + \alpha(\partial g_2) - \alpha(\partial g_1) - \alpha(\partial g_2) = 0. \qquad \square$$

The idea of the proof of Theorem 6.1 is to construct sutured manifolds which carry curves deeper and deeper in the derived series of the manifold's fundamental group, thereby allowing us to exploit this property of the Fox derivative. The construction of these curves follows the same "double-then-cut-and-paste" method we use in the proof of Theorem 5.2 to build a curve in $(\pi_1(M))^{(2)}$.

Proof of 6.1. We construct the manifolds M_K by induction on K. We make the following assumptions on M_{K-1}:

(1) The suture set γ consists of a curve γ. We realize R_+ as a closed neighborhood of g simple closed curves $c_1, \dots c_g$ disjoint away from a common basepoint;
(2) Some curve c_i has image in $\pi_1(M_{K-1})^{(K-1)} \leq \pi_1(M_{K-1})$.

Let M_1 and M_2 be two copies of M_{K-1}, and let a_1 and a_2 denote the curves from condition (2). As the sutures are single curves, there is some c_i in each with geometric intersection $i(c_i, a_j) = 1$; denote these by b_1 and b_2. We first construct an intermediate handlebody M'_K, by joining M_1 and M_2 by a one-handle $H_1 = D^2 \times D^1$ such that the disks $D^2 \times \partial D^1$ are identified with disks disjoint from all the curves c_i. Then $\pi_1(M_K) = \pi_1(M_1) * \pi_1(M_2)$. Apply the procedure from the proof of Theorem 5.2 to a_1 and a_2, as illustrated in Figure 9, to construct a curve a whose image in $\pi_1(M'_K)$ is $[a_1, a_2]$, and therefore lies in $\pi_1(M'_K)^{(K)}$. We fix a basepoint along an arc of a within H_2.

To obtain M_K, we add an additional one-handle $H_2 = D^2 \times D^1$ to M'_K by attaching the disks $D^2 \times \partial D^1$ within a small neighborhood of the basepoint, to either side of the locally separating arc of a.

To the collection of curves c_i in ∂M_K, we add a new curve c which runs around this second handle, parallel to its core, and intersecting a in exactly the basepoint. The remaining curves c_i may intersect a. We modify

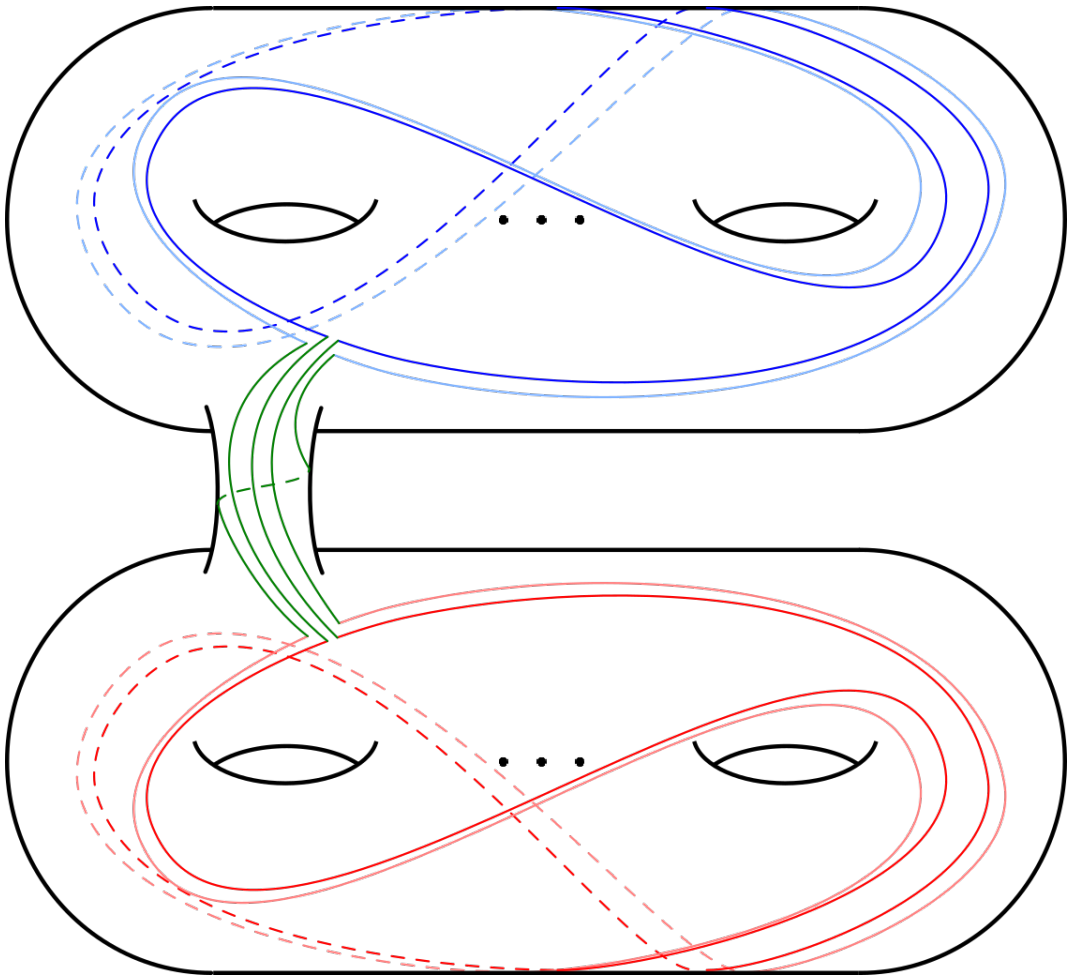


FIGURE 9. The curve a constructed from a_1 and a_2.

them as illustrated in Figure 11. Notice this procedure alters the $\pi_1(M_K)$-image of a curve in one of the following ways:

$$c_i \mapsto c_i \qquad \text{(Figure 11a)}$$
$$c_i \mapsto a_j c_i a_j^{-1} \qquad \text{(Figure 11b)}$$
$$c_i \mapsto c_i a_j^{-1} \qquad \text{(Figure 11c)}$$

These curves are once more disjoint away from a basepoint, as Figure 11d suggests. While not all combinatorial arrangements of curves are shown, the remaining cases are similar. We add one final curve $b = a_1 c a_2$, which is also included in Figure 11d, giving a total of $2g + 1$ curves. Take a closed neighborhood of these as the new R_+ and its boundary as the suture set γ defining a sutured structure on M_K. This construction shows M_K satisfies the inductive conditions (1) and (2); in particular the curve c ensures γ is connected.

To verify M_K is taut, we exhibit a sutured manifold decomposition

$$M_K \xrightarrow{S_1} M \xrightarrow{S_2} M' \xrightarrow{S_3} M'' \cup M_2,$$

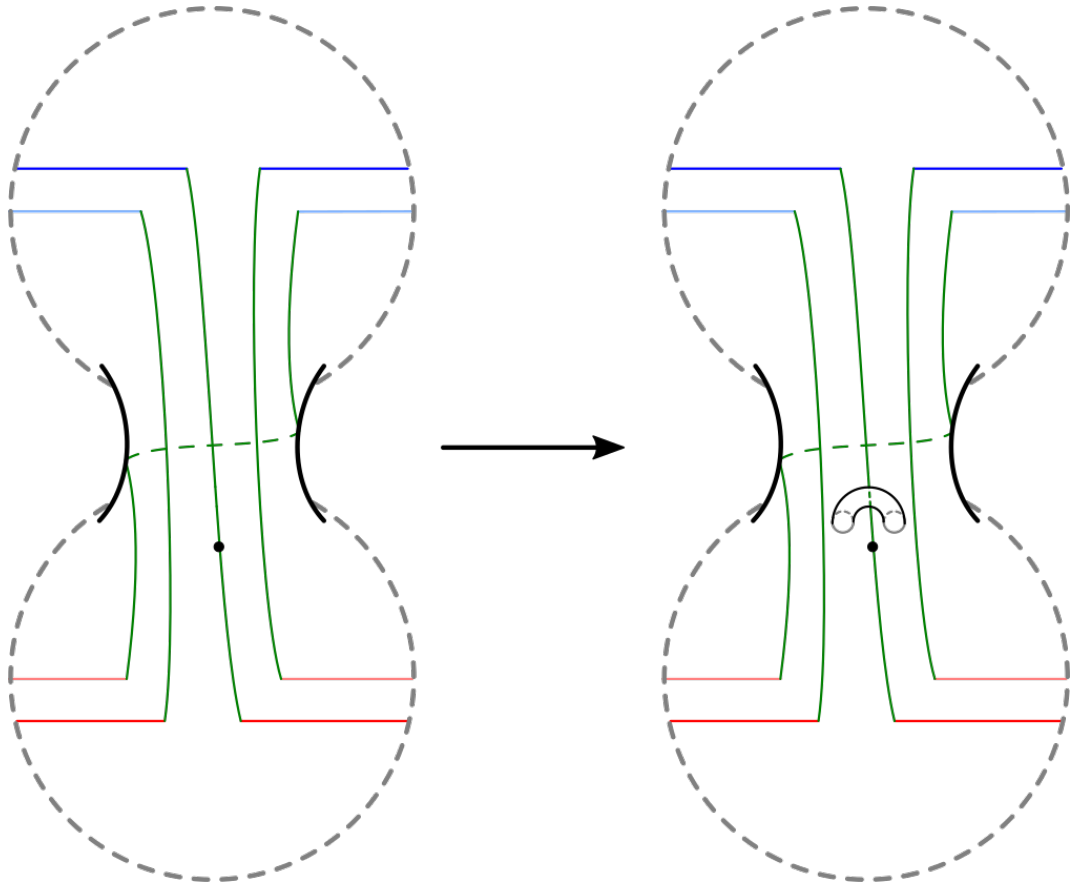


FIGURE 10. Adding the second one-handle H_2 to obtain M_K.

where M'' is another taut sutured handlebody of genus g.[2] This decomposition is illustrated in Figure 12, and described below.

The surface S_1 is the disk $D^2 \times \{\frac{1}{2}\} \subset H_2$. The decomposition kills c, and by choosing appropriate choice of orientation of S_1, the curve $b = a_1 c a_2$ becomes a_1.

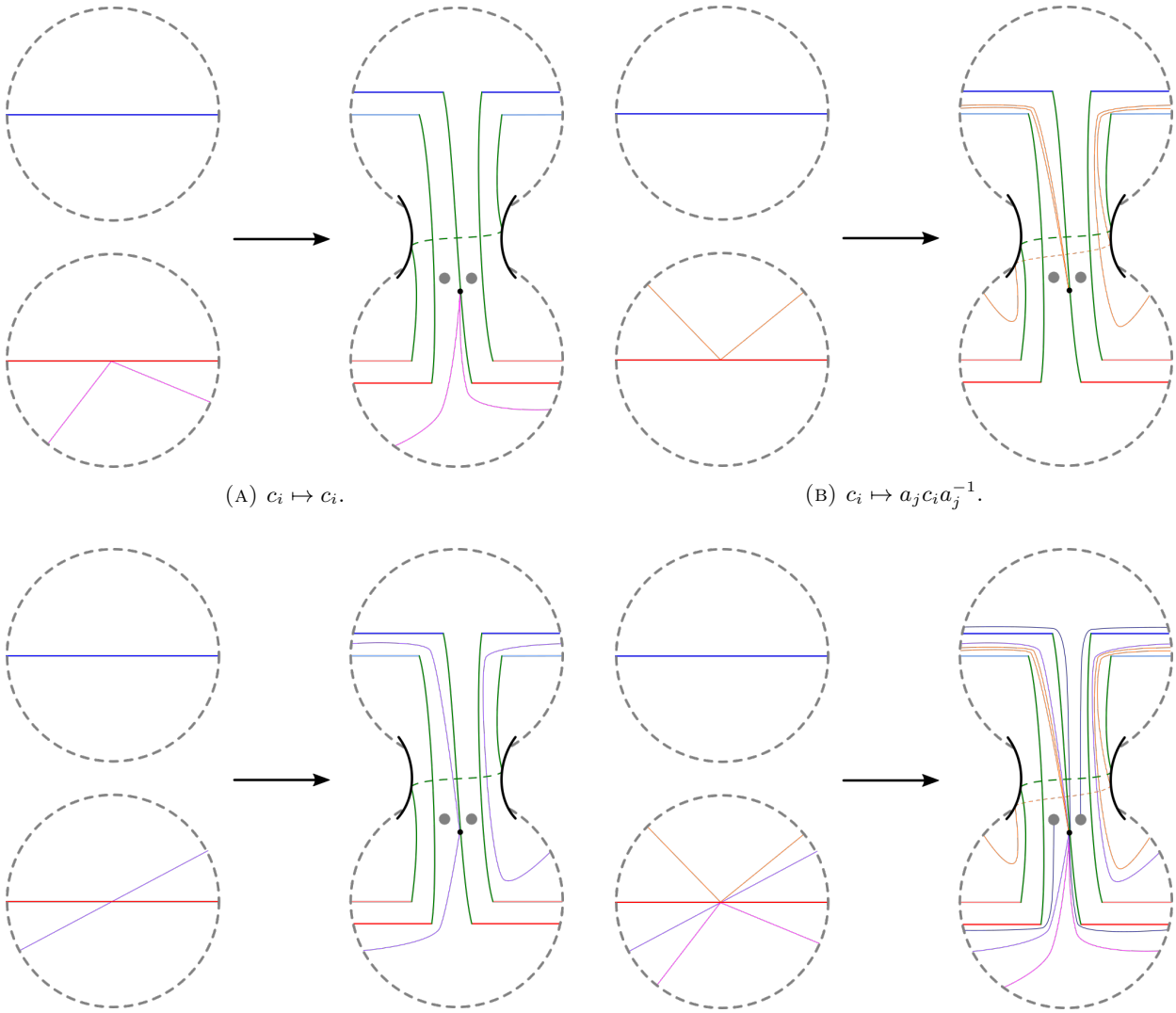
The surface S_2 is a once-punctured torus bound by the curve a. Topologically, it is the two strips between the two copies of a_1 and a_2 used to construct a, glued to the disk $D^2 \times \{\frac{1}{2}\} \subset H_1$, then pushed slightly into the handlebody. Orient S_2 so that M_2 lies on the positive side of this disk. This separates M into two genus $g+1$ handlebodies M_1' and M_2'. Notice in M_2', the two copies of a_1 used to construct a are now parallel in R_+, and similarly the copies of a_2.

Finally, S_3 consists of two disks, each cutting one of the new handles created by the decomposition along S_2. Choose these disks to be oriented to agree with a_1 and a_2, respectively. Additionally, push them off the sutures where possible, to eliminate unnecessary intersections, by dragging the disks toward the basepoint.

In M_2', this results in a disk which intersects the suture in exactly two points, cutting the a_1-bands in $R_\pm$. The remainder of the c_i are unaffected, and so the resulting sutured manifold is M_2.

In M_1', the situation is more complicated. This decomposition results in a handlebody whose sutured structure is similar to, but not exactly that of M_1. The subsurface R_+ has fundamental group with generators $c_1, \ldots, c_g$, with the exception of any curve c_i with geometric intersection $i(c_i, a_1) = 1$, such as b_1. In this

[2]In fact this shows the intermediate manifolds are also taut, in particular M, which retains the obstruction to admitting a certifying solvable representation of derived length K.



(A) $c_i \mapsto c_i$.

(B) $c_i \mapsto a_j c_i a_j^{-1}$.

(C) $c_i \mapsto c_i a_j^{-1}$.

(D)

FIGURE 11. Modifying the c_i on M_K. The handle H_2 is not shown, but is attached at the points shown.

case, b_1 is replaced by $b_1 a_1 b_1^{-1}$, and other such c_i can be replaced by $c_i b_1^{-1}$. Notice that the existence of b_1 ensures that R_+ is connected. We observe, however, that this handlebody is taut exactly when M_1 is: on the level of Fox derivatives, this difference translates to

$$\alpha(\partial i_*(b_1)) \mapsto \alpha(\partial i_*(b_1 a_1 b_1^{-1})) = \alpha(1 - i_*(b_1 a_1 b_1^{-1}))\alpha(\partial i_*(b_1)) + \alpha(i_*(b_1))\alpha(\partial i_*(a_1)),$$
$$\alpha(\partial i_*(c_i)) \mapsto \alpha(\partial i_*(c_i b_1^{-1})) = \alpha(\partial i_*(c_i)) - \alpha(i_*(c_i b_1^{-1}))\alpha(\partial i_*(b_1)).$$

In the matrix given by Proposition 4.1, this demonstrates the matrix corresponding to M'' is obtained from that for M_1 via elementary row operations. This preserves invertibility, unless $\alpha(i_*(b_1 a_1 b_1^{-1})) = 1$; in such a situation α may be perturbed away from this locus, yielding a certifying representation for both M_1 and M''.

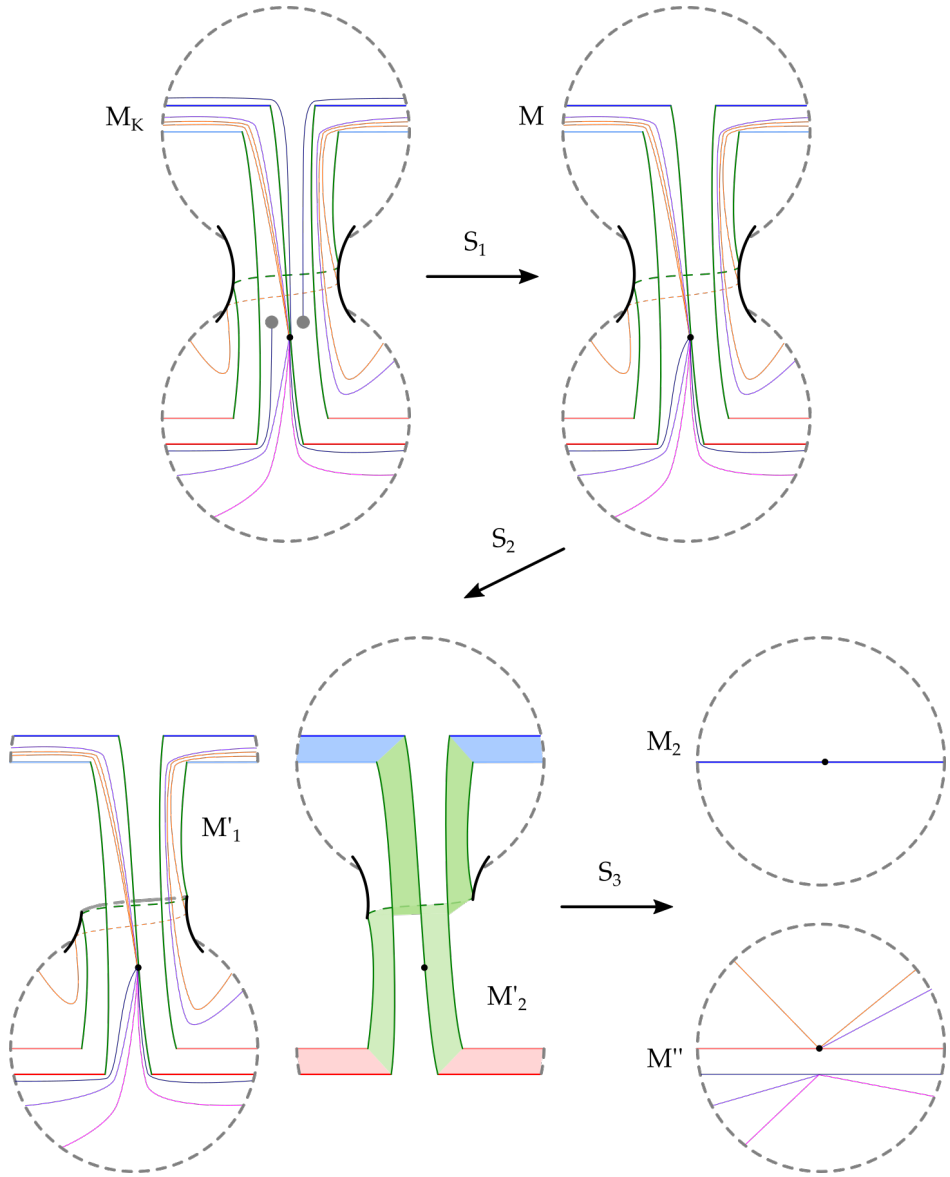


FIGURE 12. Decomposing M_K into two taut handlebodies of genus g.

Since $a \in \pi_1(M_K)^{(K)}$, by Lemma 6.3, the determinant in Proposition 4.1 vanishes for any solvable representation of degree less than K. Therefore M_K is not a twisted homology product for any such representation. $\square$

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