FAMILIES OF ALGEBRAIC CURVES AS SURFACE BUNDLES OF RIEMANN SURFACES

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1. Introduction

In this paper we study the complex structures which can occur on algebraic curves. The ideas discussed illustrate the deep tie between the algebraic objects of study in algebraic geometry, the topology of these objects when realized as surfaces, and the restrictions placed on these topological structures when viewed as complex manifolds.

An outline of the paper is as follows. Roughly the first half of the paper will study the family of complex curves $y^d = x(x-1)(x-2)$. We will show how to realize these curves as smooth Riemann surfaces and (by passing to projective space), compact Riemann surfaces. The natural question to ask is then which Riemann surfaces arise in this family. As it turns out, this information is entirely encoded in the degree of the polynomial. The crux of the first half of this paper will be in proving the degree-genus formula, which states that for a smooth complex curve whose defining polynomial has degree d, the resulting Riemann surface has genus

$$g = \frac{1}{2}(d-1)(d-2). \tag{1.1}$$

Our proof follows that given in [2].

In the second half of this paper, we expand to the more general family of smooth curves $y^d = x(x-1)(x-t)$, where t ranges over $\mathbb{C} - \{0,1\}$. For a fixed d, the degree-genus formula tells us that in fact this family can be realized as a surface bundle of genus $g = \frac{1}{2}(d-1)(d-2)$ surfaces over the thrice punctured sphere $\hat{\mathbb{C}} - \{0,1,\infty\}$. As the parameter t varies in the base space B, the complex structure on the curves varies as well. It turns out for different choices of t, these structures are non-isomorphic. Moreover, varying t around a closed loop necessarily induces a diffeomorphism on the fiber. Since the base space is not simply connected, it has nontrivial fundamental group, and if we choose a loop which is not homotopic to the identity, the resulting diffeomorphism is not isotopic to the identity (of the group of orientation-preserving diffeomorphisms on the fiber) either. This defines a map from $\pi_1(B)$ to the mapping class group of Σ_g, the group of isotopy classes of orientation-preserving diffeomorphisms on Σ_g; this map is known as the monodromy of the surface bundle. We conclude this paper by computing the monodromy of this family of curves, using the techniques in [1].

2. The curves $y^d = x(x-1)(x-2)$

In this section we study the properties of the complex algebraic curves, where C denotes the curve as a subset of $\mathbb{C}^2$ and $P \in \mathbb{C}[x,y]$ is its defining polynomial. Often it is useful to

Date: June 12, 2014.

instead consider the projectivization of C in $\mathbb{P}^2$, the solution set of the homogenization of P in $\mathbb{C}[x, y, z]$, and we will use P and C to refer to both. We focus in particular on the curves C_d defined by the polynomials $P_d(x, y) = y^d - x(x-1)(x-2)$ for $d \geq 1$.

Recall that a curve with corresponding polynomial $P \in \mathbb{C}[x, y]$ is *nonsingular* if no point $(a, b) \in \mathbb{C}^2$ satisfies

$$P(a, b) = \frac{\partial P}{\partial x}(a, b) = \frac{\partial P}{\partial y}(a, b) = 0.$$

A point which does satisfy the above three equalities is called a *singular* point of P. In the case of our curves C_d, for all $d \geq 1$, P_d is nonsingular. Note that $\frac{\partial P_d}{\partial y}(a, b) = db^{d-1} = 0$ if and only if $d > 1$ and $b = 0$. Then we must have $P_d(a, 0) = -a(a-1)(a-2) = 0$, so $a = 0$, 1, or 2. None is a solution of $\frac{\partial P_d}{\partial x}(a, 0) = -3a^2 + 6a - 2 = 0$, so P_d has no singular points.[1]

Nonsingular curves are also known as *smooth* curves, as the smooth curves are exactly those which can be realized as smooth Riemann surfaces. Such a curve over $\mathbb{C}^2$ need not be compact, but can be extend canonically to a projective curve in $\mathbb{P}^2$, which is necessarily compact.

We next illustrate how to construct a complex structure for a projective curve C, by doing so for our family of curves C_d. Let $\varphi : C_d \to \mathbb{P}^1$ be the projection $\varphi([x : y : z]) \mapsto [x : z]$. Since $[0 : 1 : 0]$ is not a point on C_d for any d, this projection is well-defined. Note that except when $x(x-z)(x-2z)z^{d-3} = 0$, the preimage $\varphi^{-1}([x : z])$ of a point in $\mathbb{P}^1$ contains exactly d points, the dth roots of $x(x-z)(x-2z)z^{d-3}$. Restricted to these points, φ is then a d-fold covering map onto its image. On all of C_d, φ is a d-fold branched cover of $\mathbb{P}^1$.

Recall for a holomorphic map $\varphi : M \to N$ between Riemann surfaces, the *ramification index* of φ at a point $p \in M$ is the (unique) positive integer $\nu_\varphi(p)$ such that for an appropriate choice of local coordinates $\psi_M : U \to \mathbb{C}$, $\psi_N : V \to \mathbb{C}$, $\psi_N \circ \varphi \circ \psi_M$ is the map $z \mapsto z^{\nu_\varphi(p)}$. In the context of algebraic curves, we give a slightly different definition of ramification index than the usual one.

Definition 2.1. Let C be an projective algebraic curve with defining polynomial $P \in \mathbb{C}[x, y, z]$. By a change of coordinates, assume $[0 : 1 : 0] \notin C$. Let $\varphi : C \to \mathbb{P}^1$ be the projection defined above. The *ramification index* $\nu_\varphi(p)$ of φ at a point $p = [a : b : c] \in C$ is the multiplicity of b as a root of the polynomial $P(a, y, c) \in \mathbb{C}[y]$.

In fact these definitions agree; both give a description of the local behavior of the map φ at a point p, which is always an m to 1 on a punctured neighborhood of p. We then define, in the usual way, the *ramification points* $\mathcal{R} \subseteq C$ to be the $p \in C$ with $\nu_\varphi(p) > 1$ and the *branch points* $\mathcal{B} \subseteq \mathbb{P}^1$ of φ to be $\varphi(\mathcal{R})$.

As $\varphi : C_d \to \mathbb{P}^1$ is a branched cover, this defines a complex structure on C_d by pulling back via φ the complex structure on $\mathbb{P}^1$. We see this more precisely as follows.

Away from the ramification points, for any point $z_0 \notin \mathcal{R}$, φ restricts to some neighborhood U of z_0 on which it a homeomorphism onto its image, and so the complex structure on $\varphi(U) \subseteq \mathbb{P}^1$ lifts to a complex structure on U. At a ramification point $z_0 \in \mathcal{R}$ with ramification index $\nu_\varphi(z_0) = m > 1$, we can find a neighborhood U of z_0 in C_d and local coordinates on $\mathbb{P}^1$ around

[1] In general, a polynomial $P(x, y) = y^d - \prod(x - a_i)^{m_i}$ is smooth exactly when $m_i = 1$ for all i; as in our example above, $\frac{\partial P}{\partial y}(a, b) = 0$ if and only if $b = 0$, so any singular point must be of the form $(a, 0)$ where a is simultaneous a root of $P(x, 0) \in \mathbb{C}[x]$ and $\frac{\partial P}{\partial x} \in \mathbb{C}[x]$. But the roots of $P(x, 0)$ are the a_i, and $(x - a_i)$ divides $\frac{\partial P}{\partial x}$ if and only if $m_i > 1$.

$\varphi(z_0)$ such that on U and in these coordinates on $\mathbb{P}^1$, φ is the map $z \mapsto z^m$. This defines local coordinates on C_d, and gives a well-defined complex structure to C_d.

Thus we have given our curve C_d the topological structure of a smooth, closed Riemann surface. It is natural then to ask which closed surface corresponds to C_d. In the next section we prove the degree-genus formula, which illustrates the deep connection between the algebra and topology of the situation, namely in the close relation between the degree of a polynomial, an algebraic property, and the genus and Euler characteristic of the curve it defines, both topological properties.

3. The degree-genus formula

In the preceding section we realized the surface C_d as a branched cover of $\mathbb{P}^1$ via the projection $\varphi([x : y : z]) \mapsto [x : z]$. A similar construction may be done for a general curve C. After a suitable change of coordinates (a diffeomorphism which preserves the genus of C), we may assume the point $[0 : 1 : 0]$ does not lie in C, so φ is well-defined. This also allows us to define a complex structure on C as the pull-back of the structure on $\mathbb{P}^1$ via φ.

We further assume that for every inflection point p of C, the tangent line $t_p(C)$ to C at p does not contain the point $[0 : 1 : 0]$. C contains only finitely many inflection points, so we can ensure this again by a suitable change of coordinates.

Remark 3.1. A point $p = [a : b : c] \in C$ has ramification index $\nu_\varphi(p) > 2$ if and only if p is an inflection point of C and $t_p(C)$ contains $[0 : 1 : 0]$. Note $\nu_\varphi(p) > 2$ if and only if

$$P(a, b, c) = \frac{\partial P}{\partial y}(a, b, c) = \frac{\partial^2 P}{\partial y^2}(a, b, c) = 0.$$

$\frac{\partial P}{\partial y}(a, b, c) = 0$ corresponds to the condition that $t_p(C)$ contains $[0 : 1 : 0]$ and $\frac{\partial^2 P}{\partial y^2}(a, b, c) = 0$ exactly when p is an inflection point.

Thus our assumption above ensures each point $p \in C$ has ramification index at most 2.

Theorem 3.2 (The degree-genus formula)**.** *Let C be a degree d nonsingular projective plane curve. Then C is a smooth surface of genus*

$$g = \frac{1}{2}(d-1)(d-2).$$

Proof. In this proof we construct an explicit triangulation of C, which allows us to compute its Euler characteristic, χ_C. This triangulation shows $\chi_C = d(3-d)$. Then, using the standard topological fact that a genus g surface has Euler characteristic $2-2g$, we see that C has genus

$$\frac{1}{2}(2-\chi_C) = \frac{1}{2}\left(2 - d(3-d)\right) = \frac{1}{2}(d-1)(d-2).$$

Consider the branched cover $\varphi : C \to \mathbb{P}^1$ defined above. Let $\mathcal{T}$ be a triangulation of $\mathbb{P}^1$ such that V contains the branch points $\mathcal{B}$ of φ; as $\mathbb{P}^1$ is homeomorphic to the sphere, construction of such a triangulation can be done easily via induction on $\#\mathcal{B}$.

Given $\mathcal{T}$, the branched cover φ allows us to lift $\mathcal{T}$ to a triangulation $\tilde{\mathcal{T}}$ of C. Away from $\varphi^{-1}(\mathcal{R}) \subseteq C$, φ is a d-sheeted cover of $\mathbb{P}^1 - \mathcal{R}$. Then each face $f \in F$ lifts to d disjoint faces in $\tilde{F}$ and similarly each edge $e \in E$ lifts to d disjoint edges in $\tilde{E}$. On $\varphi^{-1}(\mathcal{R})$, φ is not a covering map. However, for $q = [a : c] \in V$, $\varphi^{-1}(q)$ consists of the points $[a : b : c]$ such that $P(a, b, c) = 0$. $P(a, y, c)$ is a monic polynomial in y, so by the definition of $\nu_\varphi(b_i)$,

$$P(a, y, c) = \prod (y - b_i)^{\nu_\varphi([a:b_i:c])}.$$

Then $P(a, y, c)$ has exactly $d - \sum_{p=[a:b_i:c]}(\nu_\varphi(p) - 1)$ distinct roots, which correspond to the points of $\varphi^{-1}(q)$. Summing over all $p \in V$, then

$$\#\tilde{V} = d\#V - \sum_{p\in\varphi^{-1}(V)} (\nu_\varphi(p) - 1).$$

By definition, $\nu_\varphi(p) > 1$ if and only if $p \in \mathcal{R}$, so in fact

$$\#\tilde{V} = d\#V - \sum_{p\in\mathcal{R}} (\nu_\varphi(p) - 1).$$

This allows us to compute the Euler characteristic of C,

$$\begin{aligned} \chi_C &= \#\tilde{V} - \#\tilde{E} + \#\tilde{F}, \\ &= d\#V - \sum_{p\in\mathcal{R}} (\nu_\varphi(p) - 1) - d\#E + d\#F, \\ &= d\chi_{\mathbb{P}^1} - \sum_{p\in\mathcal{R}} (\nu_\varphi(p) - 1), \\ &= 2d - \sum_{p\in\mathcal{R}} (\nu_\varphi(p) - 1). \end{aligned}$$

The last step is to make sense of the quantity $\sum_{p\in\mathcal{R}}(\nu_\varphi(p) - 1)$. By our assumption on the inflection points of C, $\nu_\varphi(p) \leq 2$ for all $p \in C$. Thus $\sum_{p\in\mathcal{R}}(\nu_\varphi(p)-1) = \#\mathcal{R}$. Let the curve C' be the solution set of $\frac{\partial P}{\partial y} = 0$. The ramification points of C are intersection points of C and C'. Since C has no inflection points which lie on C', every point $p \in C \cap C'$ has distinct tangents $t_p(C)$ and $t_p(C')$ and is a nonsingular point of C'. Thus the intersection multiplicity of C and C' at p is 1. By Bézout's Theorem (see [2]), there are then exactly $\deg P \cdot \deg \frac{\partial P}{\partial y} = d(d - 1)$ ramification points. Thus

$$\chi_C = 2d - d(d - 1) = d(3 - d),$$

proving the theorem. □

We also have as a consequence of this proof a special case of the Riemann-Hurwitz formula for $\varphi : C \to \mathbb{P}^1$.

Corollary 3.3.

$$\chi_C = d\chi_{\mathbb{P}^1} - \sum_{p\in C} (\nu_\varphi(p) - 1).$$

To illustrate the proof of Theorem 3.2, we use the method given to compute the genus of one of the curves C_d.

Example 3.4. Let C be the curve defined by the polynomial $P(x, y, z) = y^4 - x(x-z)(x-2z)z$. Since $[0 : 1 : 0]$ is not a point on C, the map $\varphi : C \to \mathbb{P}^1$ is well-defined. P has degree 4, so φ is a 4-sheeted branched cover of $\mathbb{P}^1$. To compute the ramification points of φ, note that for a point $[a : b : c] \in C$, the polynomial $P(a, y, c)$ can only have a multiple root at b if both $b = 0$ and $a(a - c)(a - 2c)c = 0$. This results in four ramification points, each with ramification index 4, namely the points $[0 : 0 : 1]$, $[1 : 0 : 1]$, $[2 : 0 : 1]$, and $[1 : 0 : 0]$.

We can then realize C as the following identification space: Let U the open subset of the Riemann sphere $\hat{\mathbb{C}} \cong \mathbb{P}^1$ obtained by removing the arcs from $[0 : 1]$ to $[1 : 1]$, $[1 : 1]$ to $[2 : 1]$, and $[2 : 1]$ to $[1 : 0]$. By opening up the removed arcs, we see U is homeomorphic to a disk, and in particular, the interior of a hexagon with edges corresponding to the arcs and vertices corresponding to the branch points. U is simply connected and contains none of the branch points of φ, so $\varphi^{-1}(U)$ is the disjoint copy of four open sets, each homeomorphic to the interior of a hexagon, which map homeomorphically onto U. We then construct C by appropriately identifying the edges of these hexagons, shown in Figure 1 below.

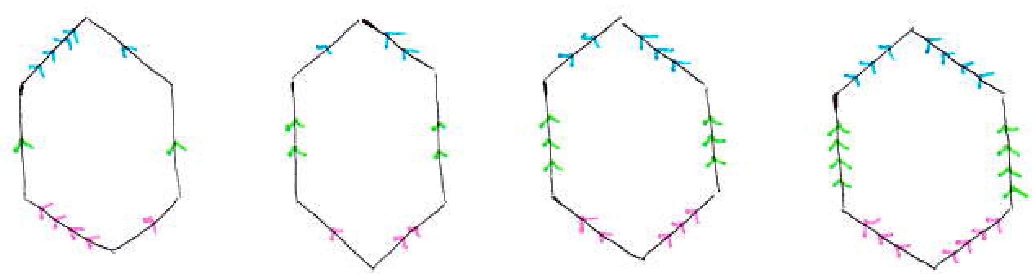


FIGURE 1

A triangulation of U lifts to the same triangulation on each hexagon in $\varphi^{-1}(U)$, which then glues together to a triangulation of C, illustrated in Figure 2.

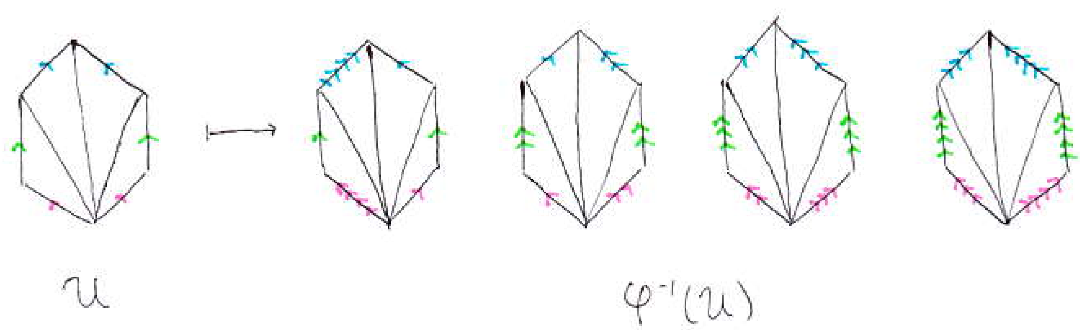
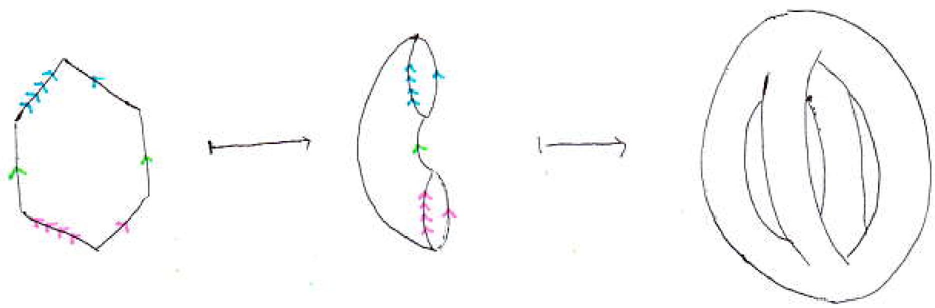


FIGURE 2

This shows

$$\chi_C = \#\tilde{V} - \#\tilde{E} + \#\tilde{F} = 4 - 24 + 16 = 4(\chi_{\mathbb{P}^1}) - 4(4-1) = -4$$

Thus C has Euler characteristic -4, and so the genus of C is $\frac{1}{2}(2-(-4)) = 3$, which agrees with the degree-genus formula. Additionally, by applying the identifications on the hexagons above, we can visualize C in $\mathbb{R}^3$ in the following way.



4. A family of curves as a holomorphic family

In this section we introduce a generalization of our curves C_d from the previous sections. Consider the family

$$E = \{([x:y:z],t) \mid y^d = x(x-z)(x-tz)z^{d-3}\} \subseteq \mathbb{P}^2 \times \hat{\mathbb{C}}, \tag{4.1}$$

When $t \neq 0, 1, \infty$, each curve $y^d = x(x-z)(x-tz)z^{d-3}$ is nonsingular, by the argument in Footnote 1. Then, by the degree-genus formula (Theorem 3.2), each curve is a smooth, closed surface of genus $g = \frac{1}{2}(d-1)(d-2)$. Let $B = \hat{\mathbb{C}} - \{0, 1, \infty\}$. Then $p : E \to B$, the map $([x:y:z],t) \mapsto t$, defines a surface bundle $\Sigma_g \to E \to B$.

Each fiber $p^{-1}(t)$ of $p : E \to B$ carries a complex structure, constructed in the same way as for $p^{-1}(2) = C_d$ in Section 2, and these structures vary holomorphically with t. Thus the total space E can be given the structure of a complex 2-manifold. A surface bundle which satisfies these condtions is called a *holomorphic family of Riemann surfaces.*

While any two fibers of such a family are diffeomorphic to each other, they are not necessarily isomorphic as Riemann surfaces; isomorphism in this sense is a much stronger condition. This richer structure of holomorphic families leads to the following definitions.

Definition 4.2. If each point $t \in B$ has a neighborhood U such that $p : p^{-1}(U) \to U$ is isomorphic to $p : p^{-1}(t) \times U \to U$ as holomorphic families, then $p : E \to B$ is *locally trivial.* If we assume B is connected, this condition implies any two fibers $p^{-1}(t)$ and $p^{-1}(s)$ are isomorphic as Riemann surfaces. If $p : E \to B$ is not locally trivial, we say it is *truly varying.*

Example 4.3. Let $p : E \to B$ be the holomorphic family defined in (4.1). The complex structure of each fiber $p^{-1}(t) = C_d$ is the pullback by φ of that on $\mathbb{P}^1$, where $\varphi : C_d \to \mathbb{P}^1$ is a d-fold branched cover over $\{0, 1, \infty, t\}$. Then if $p^{-1}(t)$ and $p^{-1}(s)$ are isomorphic as

Riemann surfaces, their structures are lifted from isomorphic structures on $\mathbb{P}^1 - \{0, 1, \infty, t\}$ and $\mathbb{P}^1 - \{0, 1, \infty, s\}$, respectively.

The space of complex structures on $\mathbb{P}^1$ (or equivalently, $\hat{\mathbb{C}}$) with four removed points, up to biholomorphic equivalence, is parametrized by $t \in B$, since any triple of points $p, q, r \in \mathbb{P}^1$ can be taken biholomorphically to the triple $0, 1, \infty$ by a unique Möbius transformation. Thus in fact $t = s$, so $p^{-1}(t)$ and $p^{-1}(s)$ are the same fiber. This shows our family $p : E \to B$ is an example of a truly varying holomorphic family.

5. The mapping class group and the monodromy of a surface bundle

In the last example of Section 4, we observe that as t varies in the base space B, the complex structure on the fiber $p^{-1}(t)$ varies as well. One might ask what happens to the complex structure on the fiber when t travels around a closed loop in B, and in particular, a closed loop which is not null-homotopic. It turns out this determines a diffeomorphism on $p^{-1}(t)$, which, up to isotopy, is uniquely determined by the homotopy class of the loop in B. In this last section we will study which such isotopy classes of diffeomorphisms can arise in this way, specifically in the case of two holomorphic families.

We begin with the following definition.

Definition 5.1. Let S be a smooth surface. Then the *mapping class group* $\mathrm{Mod}(S)$ is the group of isotopy classes of orientation-preserving diffeomorphisms of S (which fix ∂S).

The mapping class group is a topological invariant and an object of ubiquity in surface topology. Although elements of $\mathrm{Mod}(S)$ are isotopy classes of diffeomorphisms, we will often refer to them by specific diffeomorphisms, and it is understood the isotopy class is meant. The process described above gives a well-defined homomorphism $\rho : \pi_1(B, t) \to \mathrm{Mod}(p^{-1}(t))$. When B is connected, ρ is independent of choice of t, so if $p^{-1}(t)$ is a genus g surface, we have a homomorphism $\rho : \pi_1(B) \to \mathrm{Mod}(\Sigma_g)$.

Definition 5.2. The homomorphism $\rho : \pi_1(B) \to \mathrm{Mod}(\Sigma_g)$ is called the *monodromy* of the holomorphic family $\Sigma_g \to E \to B$.

Example 5.3. Consider the holomorphic family

$$E = \{y^2 = (x-1)(x^2-t) \mid (x, y, t) \in \mathbb{C}^2 \times B\},$$

where $B = \hat{\mathbb{C}} - \{0, 1, \infty\}$ and $p : E \to B$ is given by $p(x, y, t) = t$. By the degree-genus formula (Theorem 3.2), the fibers of this family are genus 1 Riemann surfaces.

As the fundamental group $\pi_1(B)$ is the free group on two generators (namely the loops σ and τ in Figure 3, based at 1/2), in order to compute $\rho : \pi_1(B) \to \mathrm{Mod}(T^2)$, it suffices to determine the images of these two loops.

For $t \in B$, $p^{-1}(t)$ is a two-sheeted branched cover of $\hat{\mathbb{C}} - \{1, \infty, \sqrt{t}, -\sqrt{t}\}$, which is realized by the hyperelliptic involution h shown in Figure 4.

This branched cover lifts to a fiberwise branched cover of our bundle $p : E \to B$ over the bundle $p' : E' \to B$, where a fiber of E' is the sphere with four marked points, which we will denote just by S^2. Via h, any diffeomorphism α in the image of $\rho' : \pi_1(B) \to \mathrm{Mod}(S^2)$

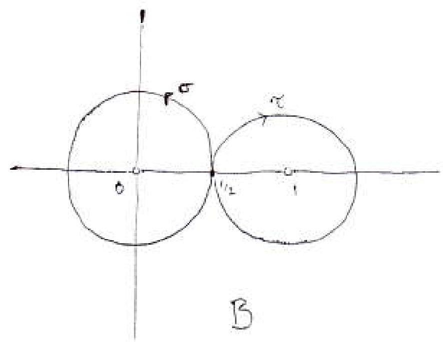


FIGURE 3

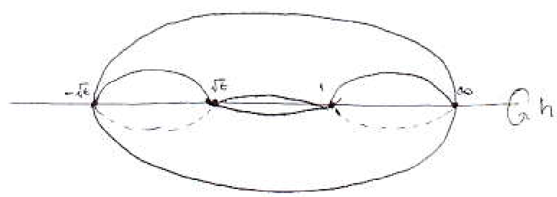


FIGURE 4

corresponds to a diffeomorphism $\tilde{\alpha}$ of T^2 such that the diagram

$$\begin{array}{ccc} T^2 & \xrightarrow{\tilde{\alpha}} & T^2 \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle h} \\ S^2 & \xrightarrow{\alpha} & T^2 \end{array}$$

commutes, and vice versa. As h itself is a diffeomorphism of T^2, and therefore an element of $\mathrm{Mod}(T^2)$, this means that every $\tilde{\alpha}$ in the image of ρ must have an isotopy-equivalent representative which commutes with h. Every diffeomorphism of T^2 which commutes with h descends to a diffeomorphism of S^2, so we can define a map $\pi : C(h) \to \mathrm{Mod}(S^2)$, where $C(h)$ denotes the centralizer of h in $\mathrm{Mod}(T^2)$. The only nontrivial diffeomorphism in $C(h)$ which lies in the kernel of π is the hyperelliptic involution h itself, so we have an exact sequence

$$1 \longrightarrow \langle h \rangle \longrightarrow C(h) \xrightarrow{\pi} \mathrm{Mod}(S^2).$$

This gives the factorization $\rho' = \pi \circ \rho$. Thus once we determine the monodromy ρ', it is just a matter of lifting via h the diffeomorphisms in the image of ρ' to ones on T^2 to determine the monodromy ρ.

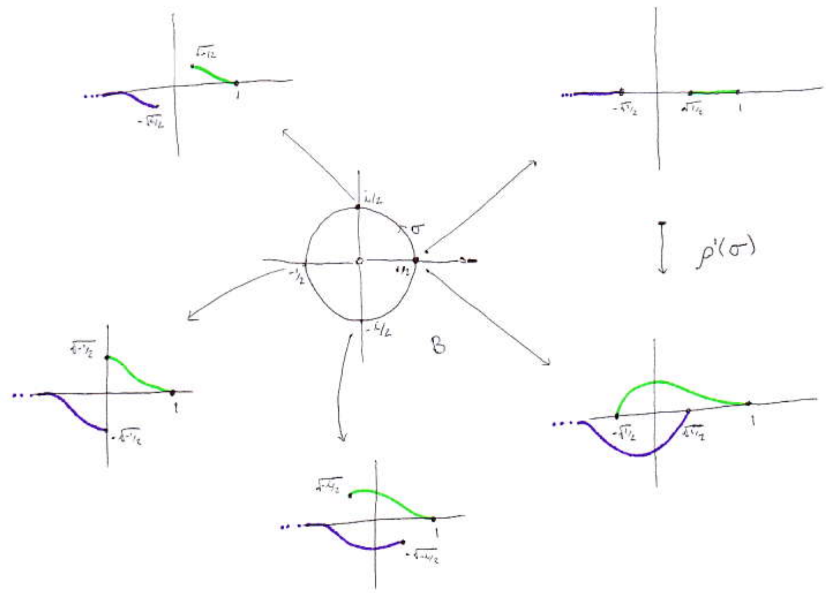


FIGURE 5

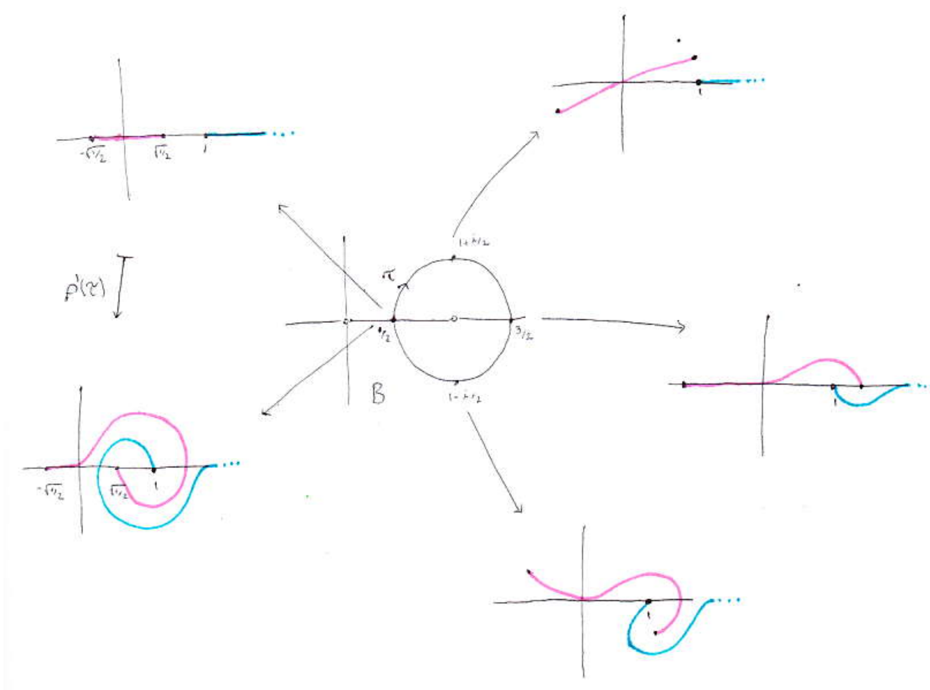


FIGURE 6

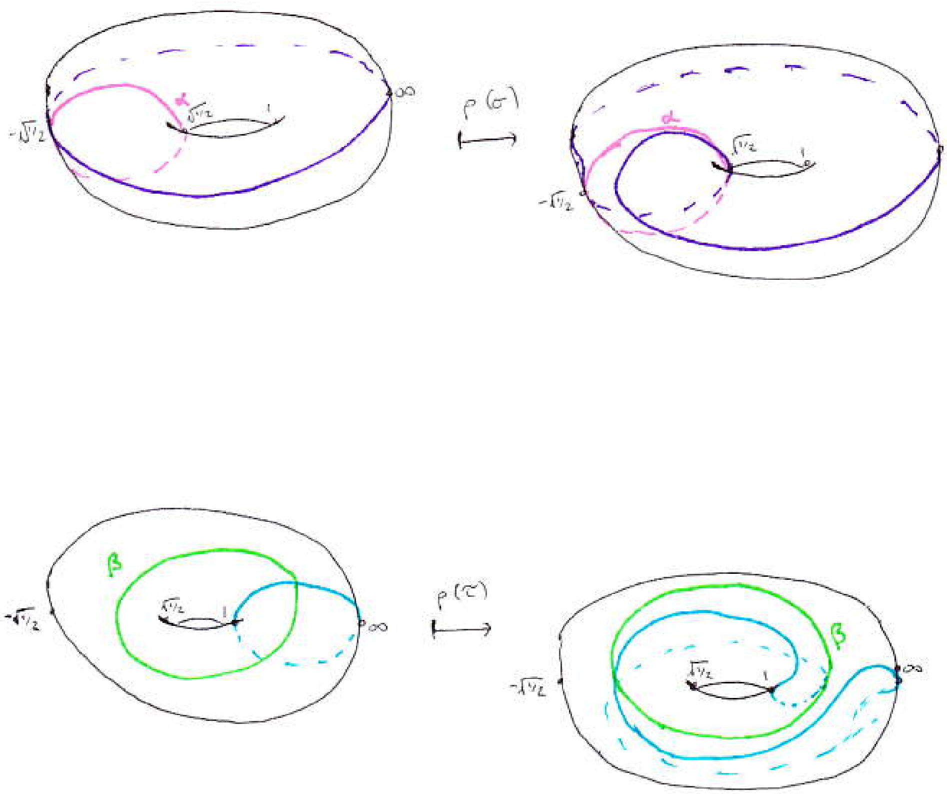


FIGURE 7

To compute ρ', we just need to determine $\rho'(\sigma)$ and $\rho'(\tau)$. Figures 5 and 6 illustrate the diffeomorphisms of $\hat{\mathbb{C}} - \{1, \infty, \sqrt{1/2}, -\sqrt{1/2}\}$ determined by σ and τ. We see that $\rho'(\sigma)$ is a diffeomorphism which interchanges $\sqrt{1/2}$ and $-\sqrt{1/2}$ while fixing 1 and ∞ and $\rho'(\tau)$ is a diffeomorphism which fixes 1 and ∞ and moves $\sqrt{1/2}$ around 1 once.[2]

Finally, it remains to find lifts of $\rho'(\sigma)$ and $\rho'(\tau)$ in $\mathrm{Mod}(T^2)$, namely elements of $\mathrm{Mod}(T^2)$ which cover these diffeomorphisms under the branched cover $T^2 \to S^2$. These are illustrated in Figure 7: $\rho(\sigma)$ is a Dehn twist around the curve α and $\rho(\tau)$ is the square of a Dehn twist around β.

We conclude this paper by computing the monodromy for the holomorphic family (4.1) for the case $d = 4$.

[2] These mapping classes are commonly referred to a half twist and a whole twist, respectively. The reasoning for this naming comes from the canonical identification of the mapping class group of $\hat{\mathbb{C}} - \{1, \infty, \sqrt{1/2}, -\sqrt{1/2}\}$ with the braid group on three strands, with strands corresponding to 1, $\sqrt{1/2}$, and $-\sqrt{1/2}$; a half twist (respectively, whole twist) is an operation on the braid which interchanges two adjacent strands once (respectively, twice).

Example 5.4. Let $B = \hat{\mathbb{C}} - \{0, 1, \infty\}$ and let

$$E = \{([x : y : z], t) \mid y^4 = x(x - z)(x - tz)z\},$$

with $p : E \to B$ defined by $p([x : y : z], t) = t$. By the degree-genus formula (Theorem 3.2), a fiber $p^{-1}(t)$ is a genus 3 surface.

The fundamental group of B, which is the same as in the preceding Example 5.3, is freely generated by the loops σ and τ based at 1/2. Then to compute the monodromy $\rho : \pi_1(B) \to \mathrm{Mod}(\Sigma_3)$ we just need to find the images of σ and τ.

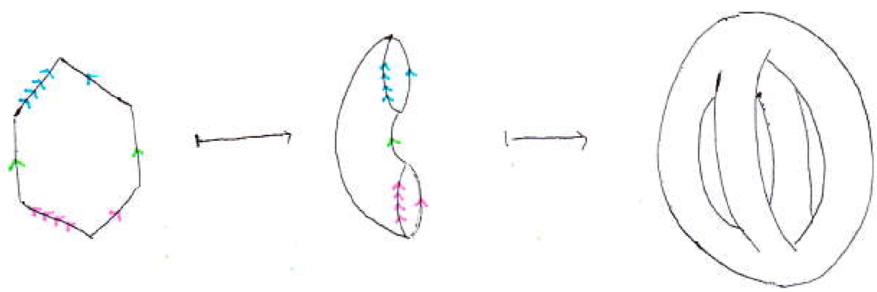


FIGURE 8

In Example 3.4, we constructed the fiber $p^{-1}(2) = \Sigma_3$ as the identification space of four hexagons with identified edges. A similar construction may be done for every fiber $p^{-1}(t)$. This shows each fiber is a four-sheeted branched cover over S^2 with branch points $\{0, 1, \infty, t\}$, which is attained by the order 4 diffeomorphism h illustrated in Figure 8.

As in Example 5.3, this branched cover allows us to lift diffeomorphisms of S^2 which (setwise) fix $\{0, 1/2, 1, \infty\}$ to ones of $p^{-1}(1/2) = \Sigma_3$ using h. So we begin by computing the monodromy $\rho' : \pi_1(B) \to \mathrm{Mod}(S^2)$.

Figure 9 shows that the loop σ corresponds to a diffeomorphism of S^2 which moves the point 1/2 around 0 once and fixes 1 and ∞, a whole twist exchanging 0 and 1/2. Analogously, τ induces a diffeomorphism which moves 1/2 around 1 once and fixes 0 and ∞, a whole twist exchanging 1/2 and 1.

Finally, Figure 10 illustrates how to use h to lift $\rho'(\sigma)$ to a diffeomorphism of Σ_3, the product of Dehn twists T_{α_1}, T_{α_2}, T_{α_3}, and T_{α_4}. $\rho'(\tau)$ can be lifted similarly. These then determine the monodromy $\rho : \pi_1(B) \to \mathrm{Mod}(\Sigma_3)$.

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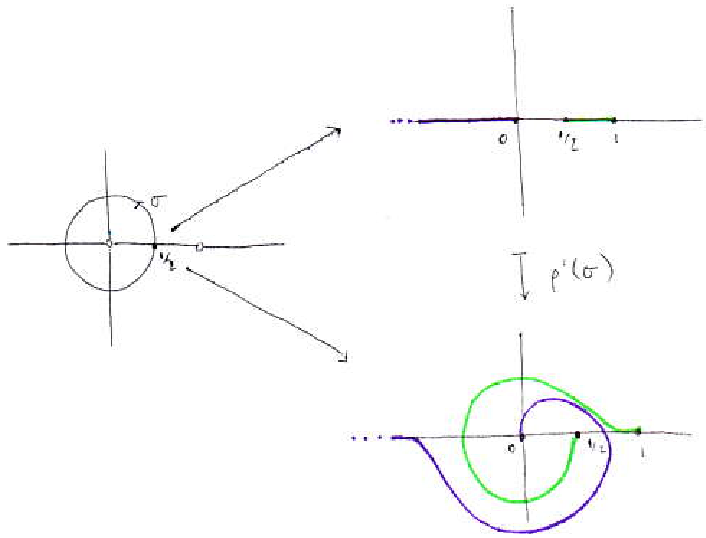

FIGURE 9

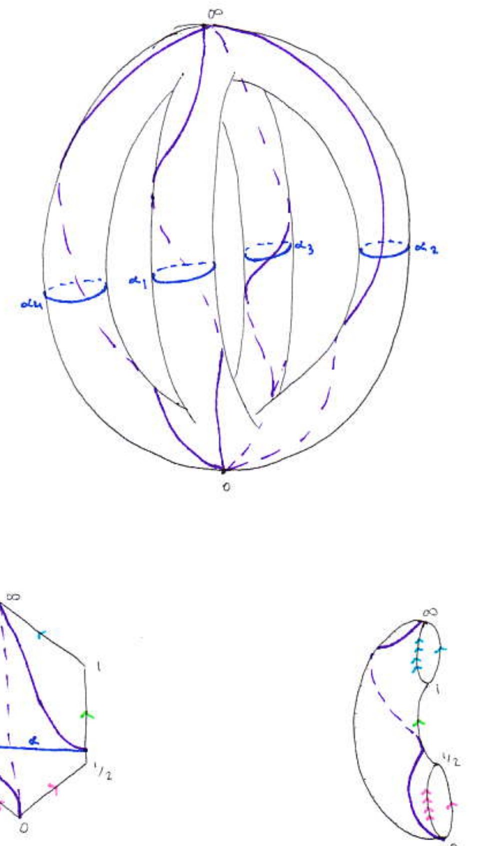

FIGURE 10