Surface embeddings in $\mathbb{R}^3$ via the lens of the crease set

Margaret Nichols
Joint with William Menasco
University at Buffalo
25 June 2021

Crease set

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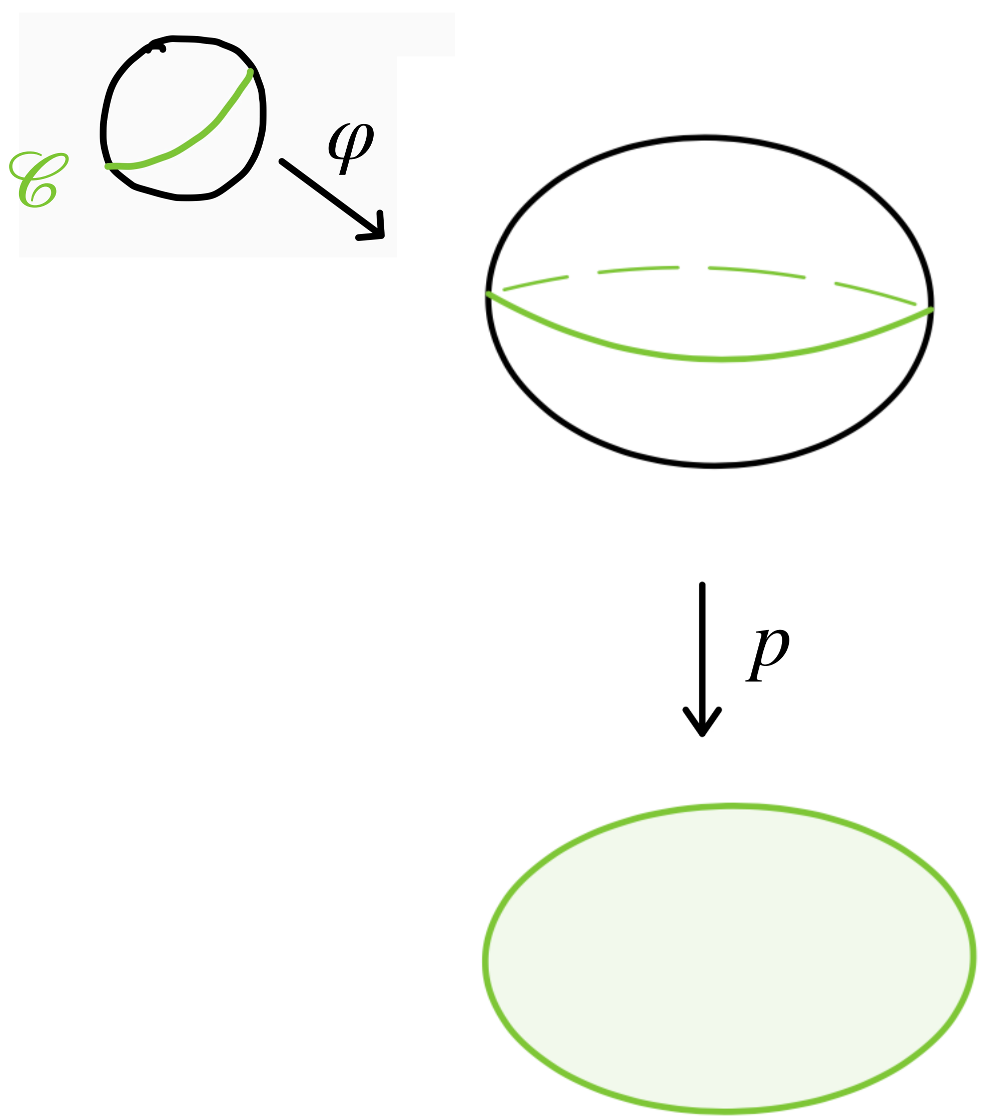

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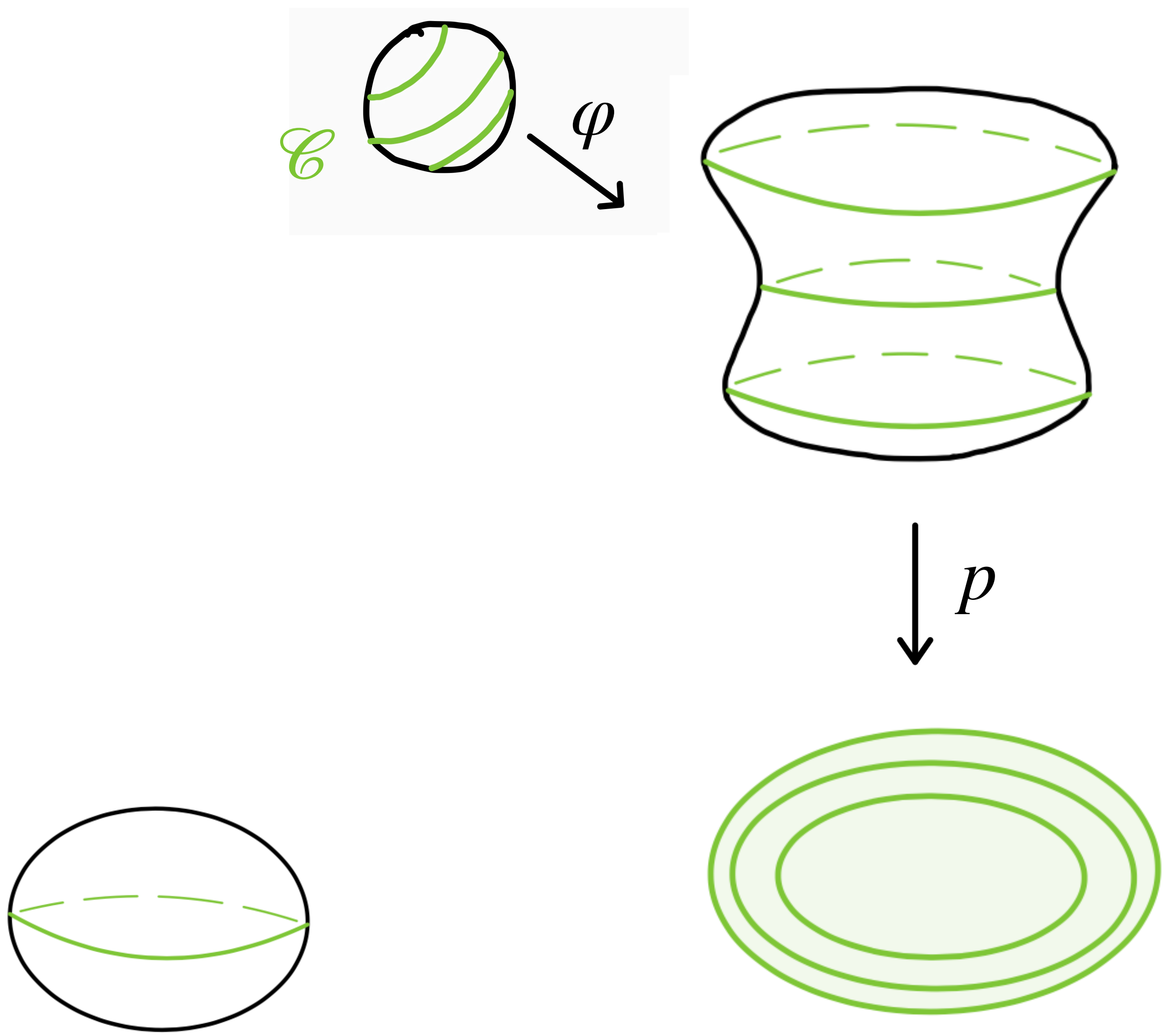

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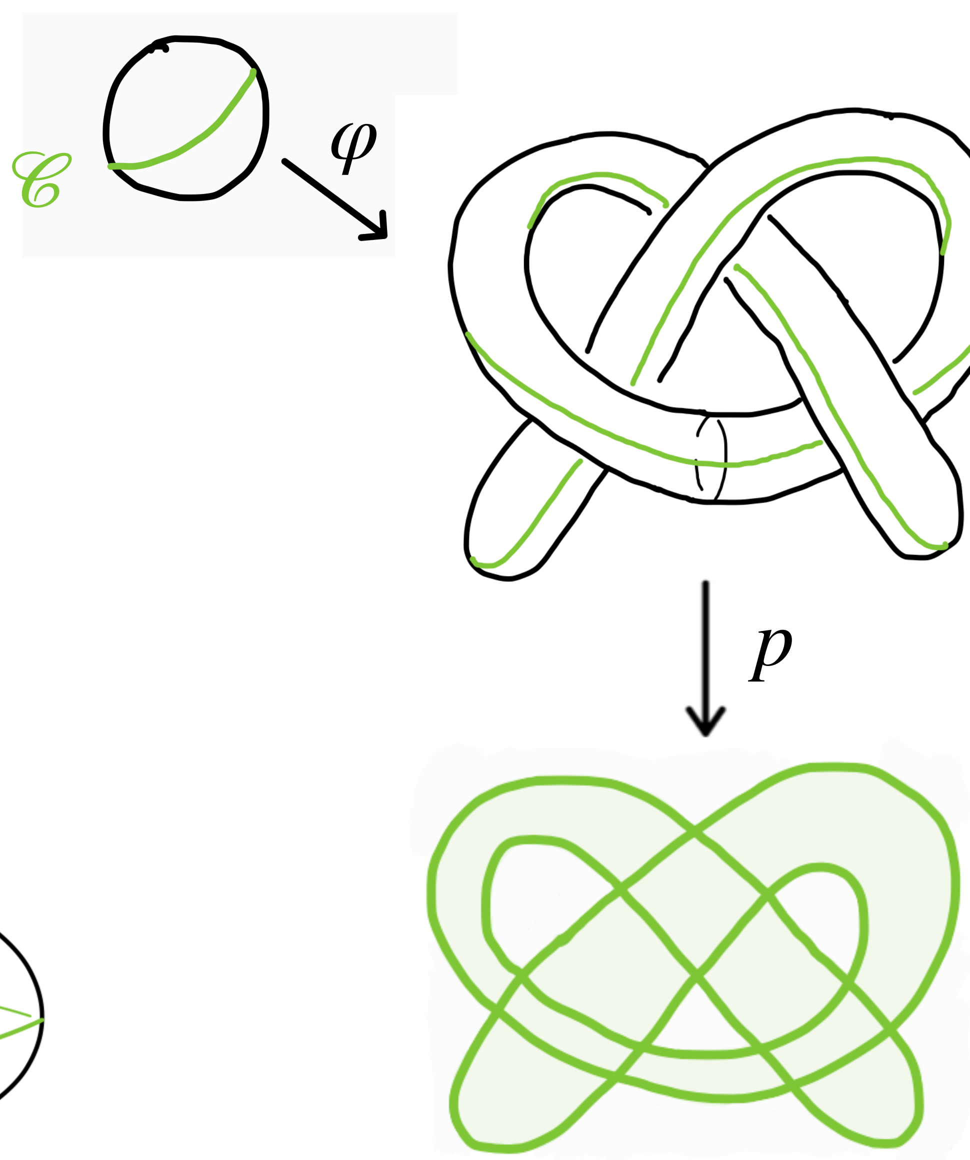

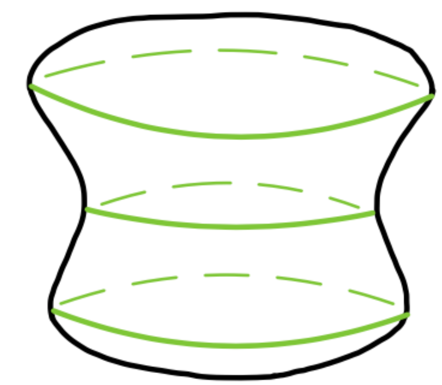

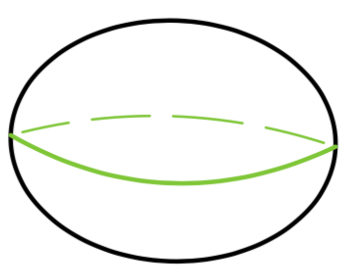

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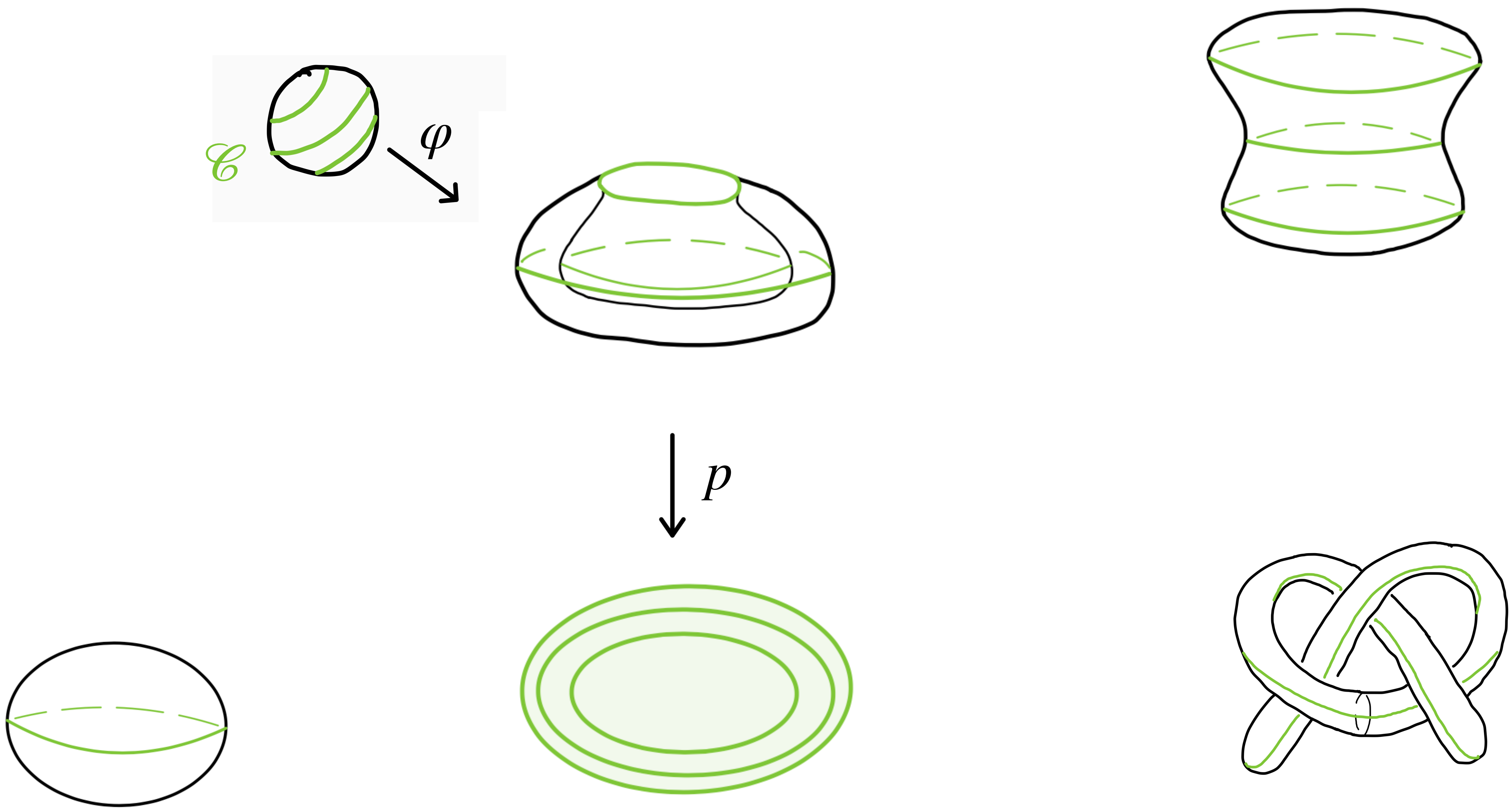

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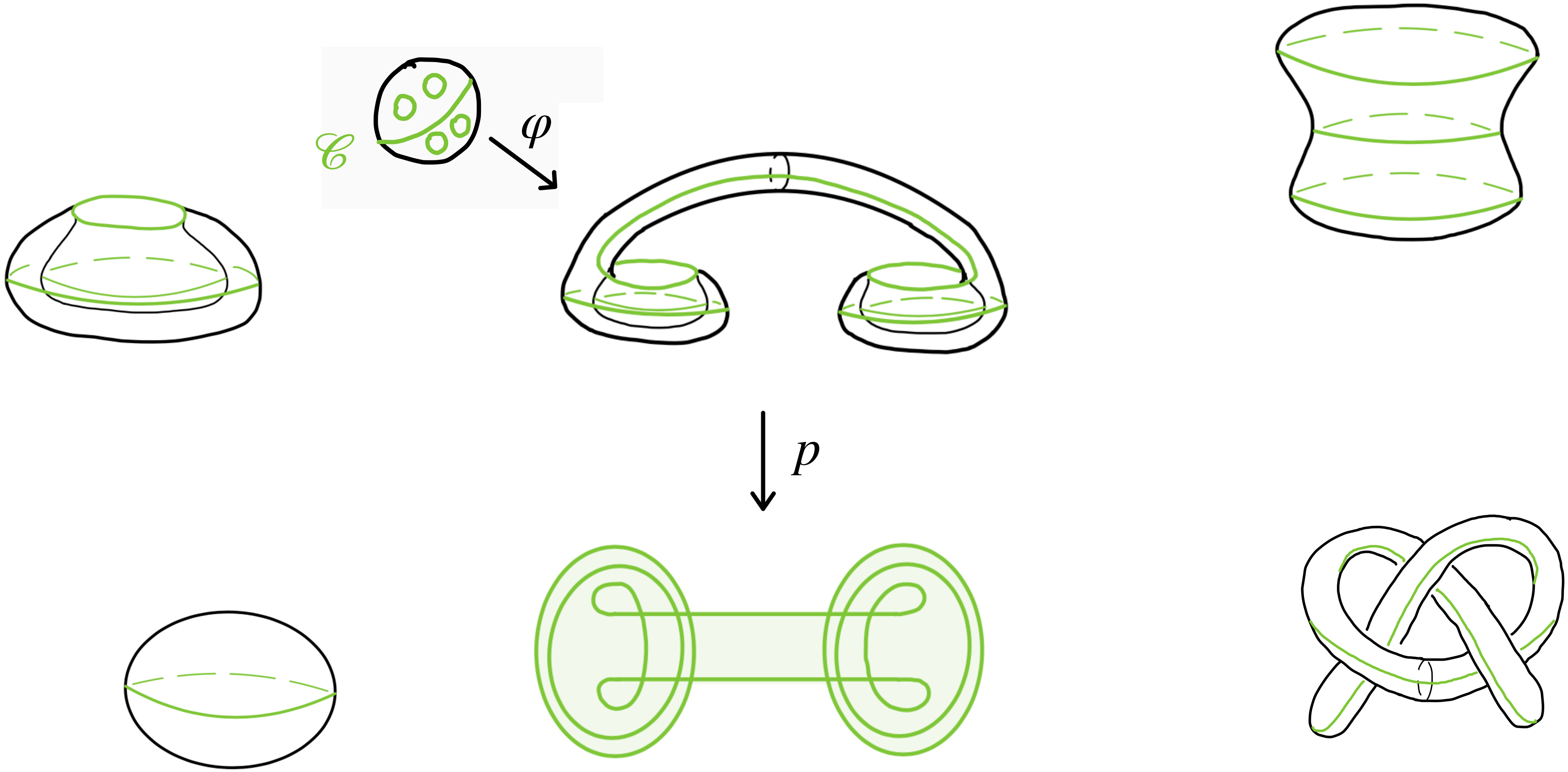

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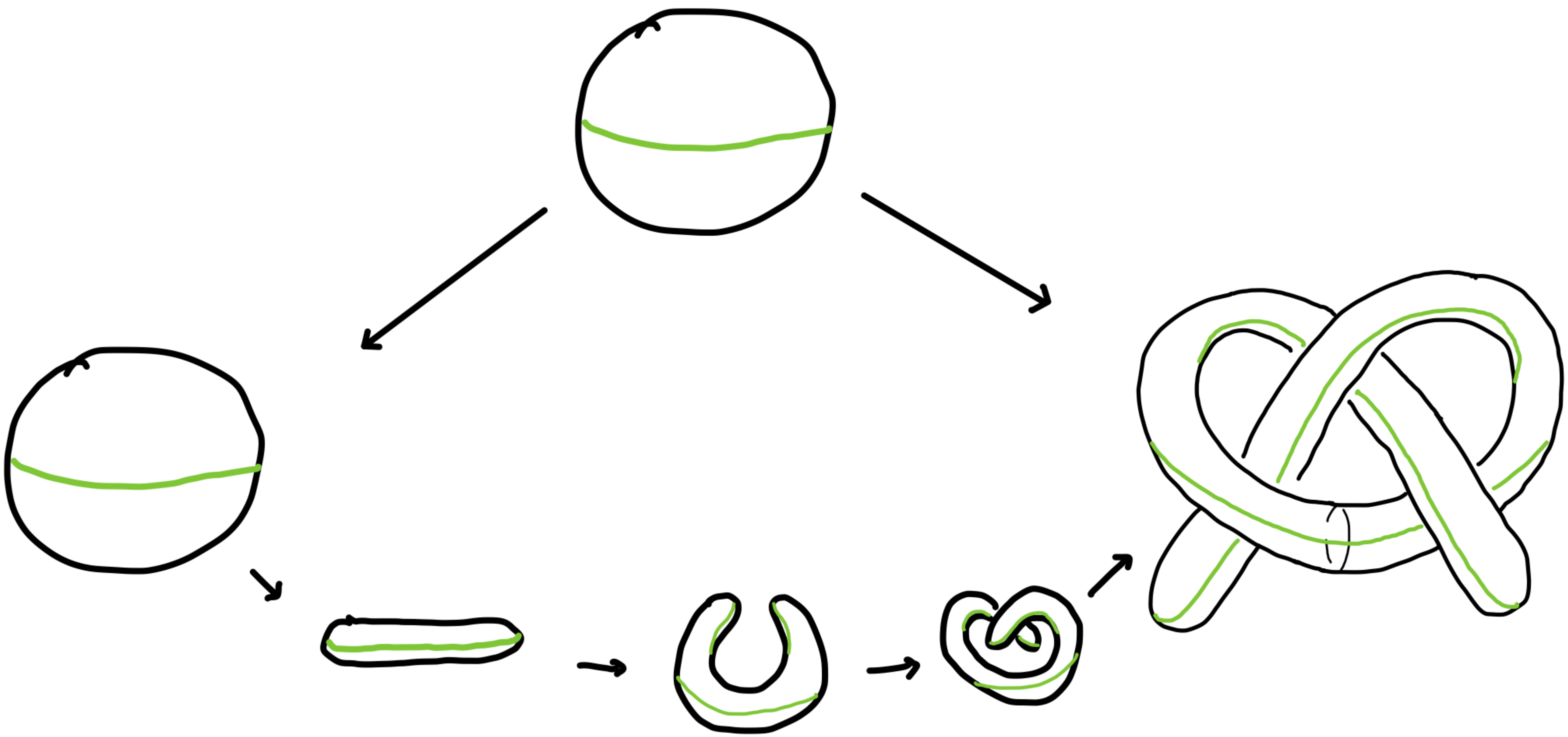

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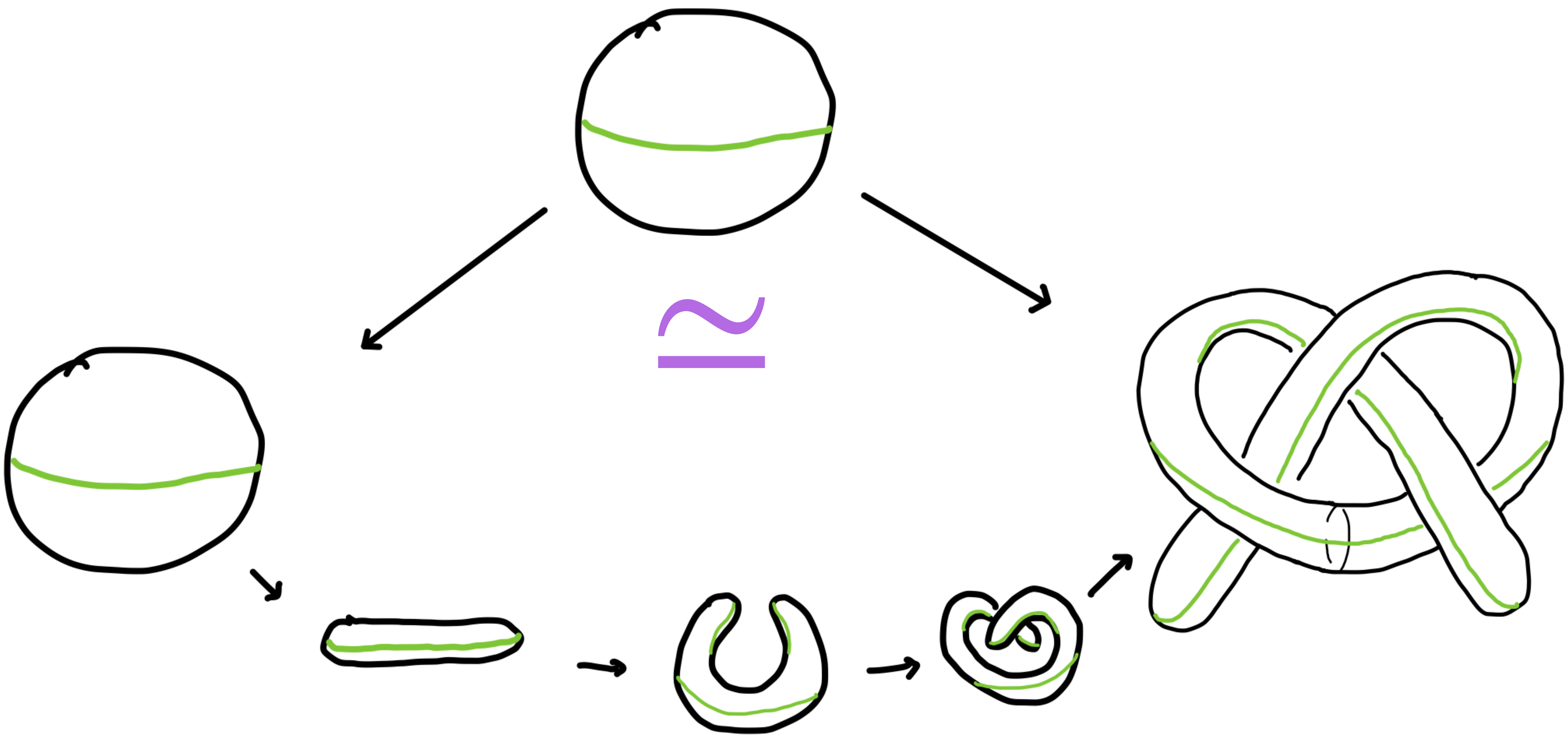

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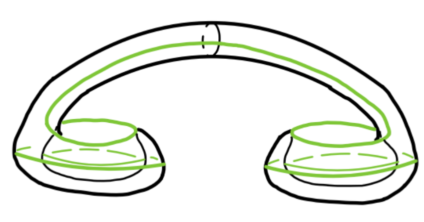

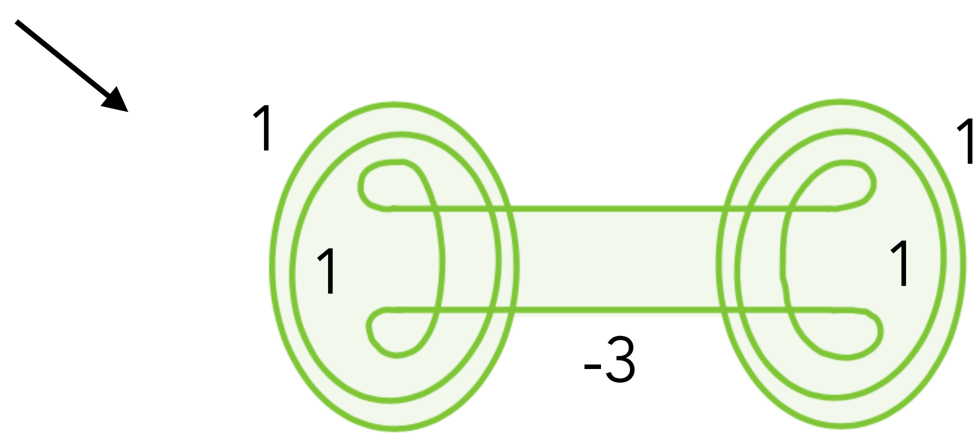

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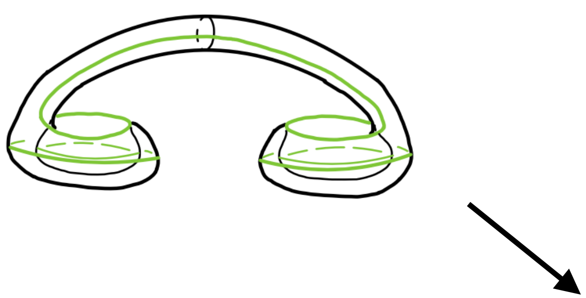

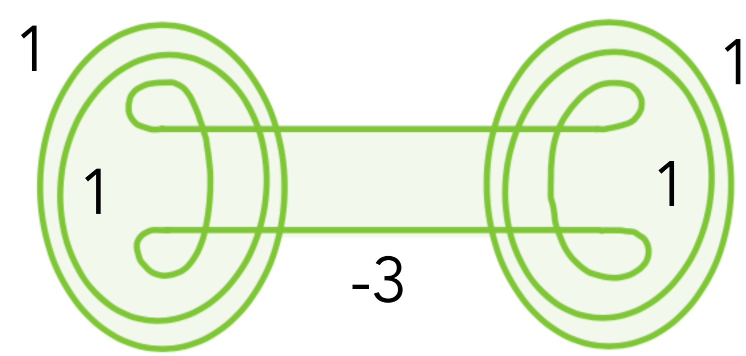

Turning number is *preserved* by regular isotopy!

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In particular,

$$\sum_{\gamma \in \mathscr{C}} t(\gamma) = 1 \qquad \text{and} \qquad \sum_{\gamma \in \partial S} t(\gamma) = \chi(S)$$

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Idea: Gauss–Bonnet

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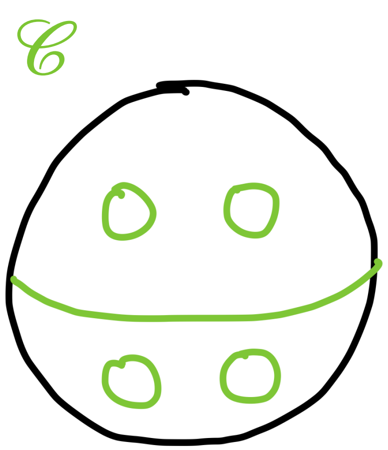

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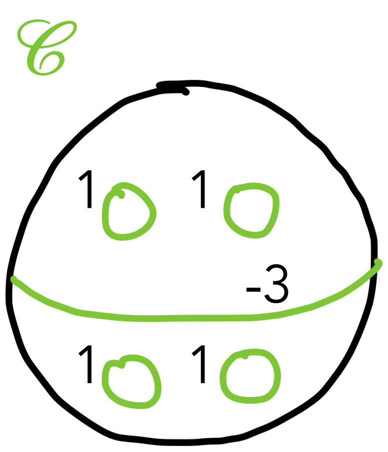

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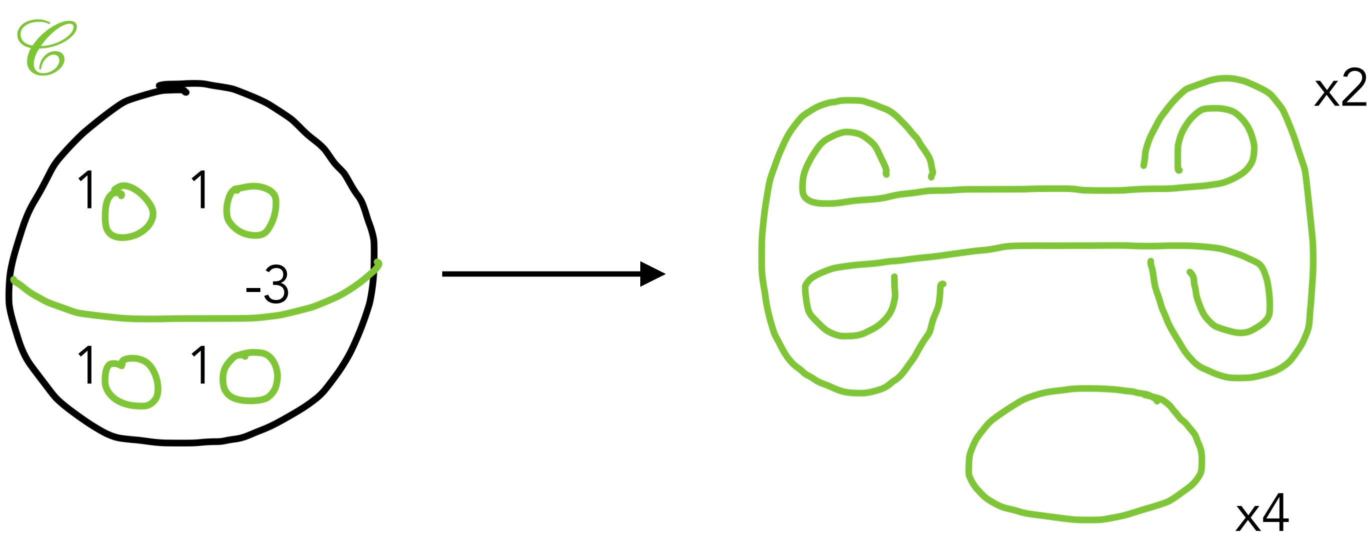

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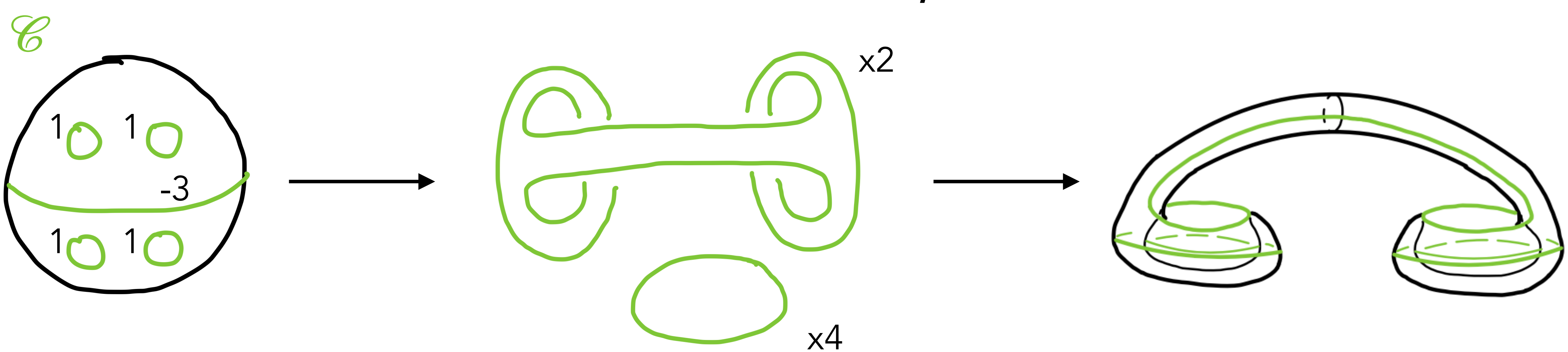

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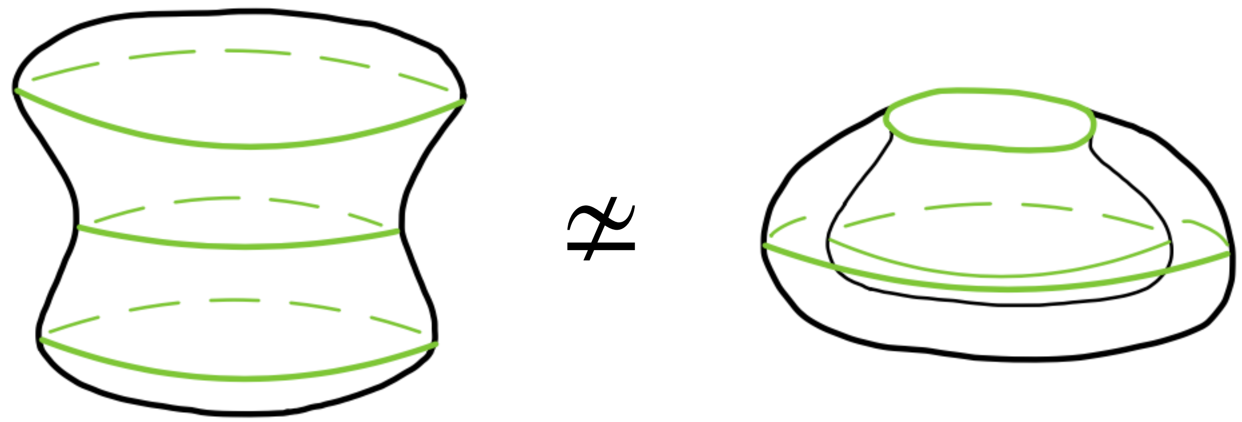

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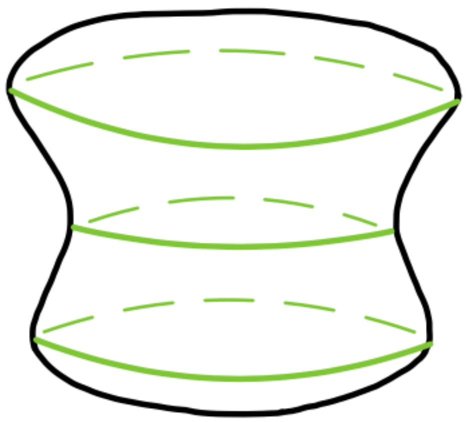
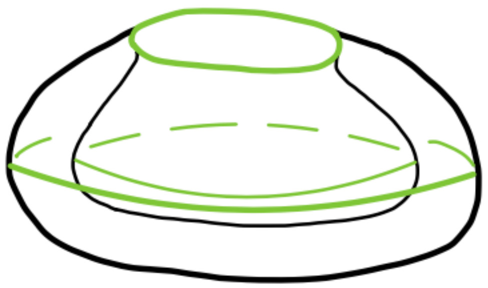


$\not\simeq$



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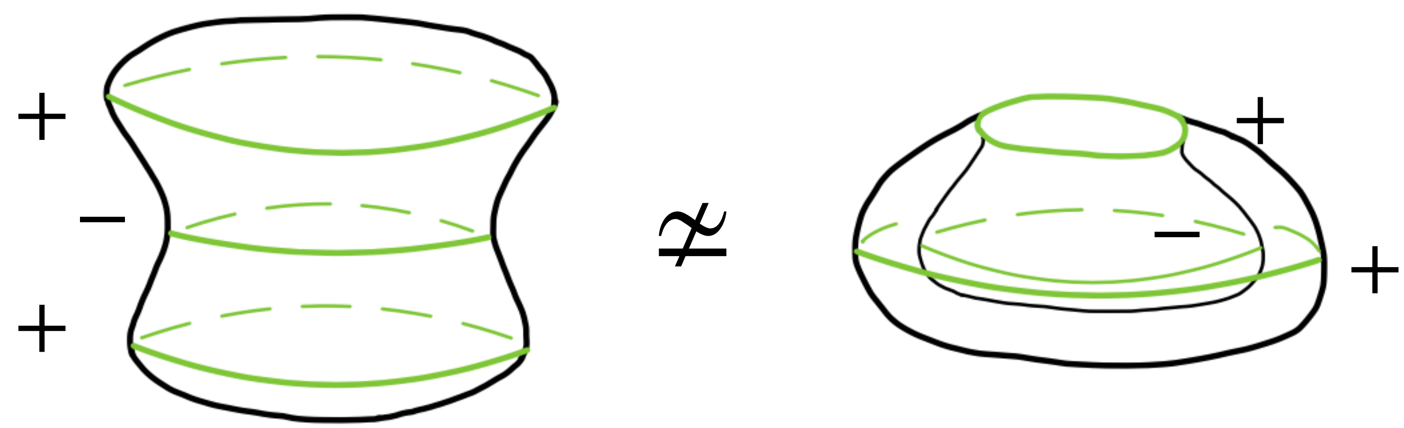

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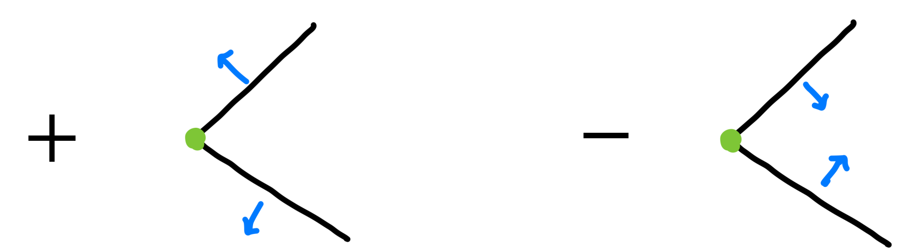

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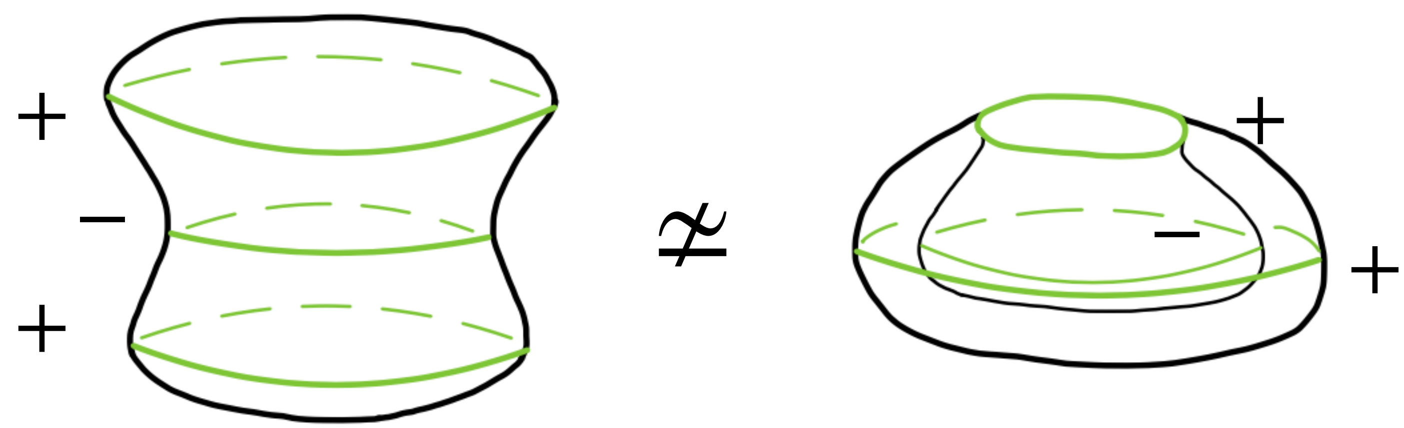

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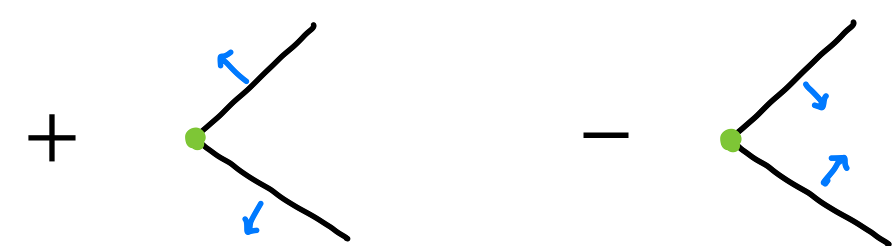

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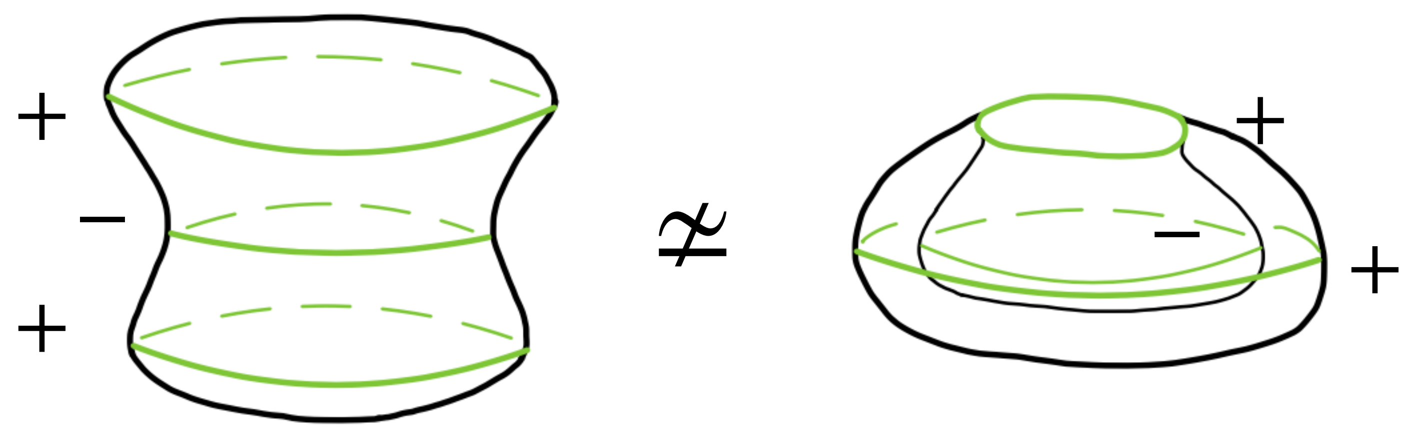

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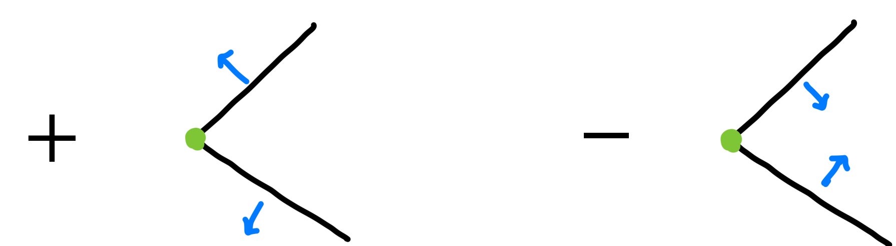

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➡ Need to understand constraints from folding orientations.

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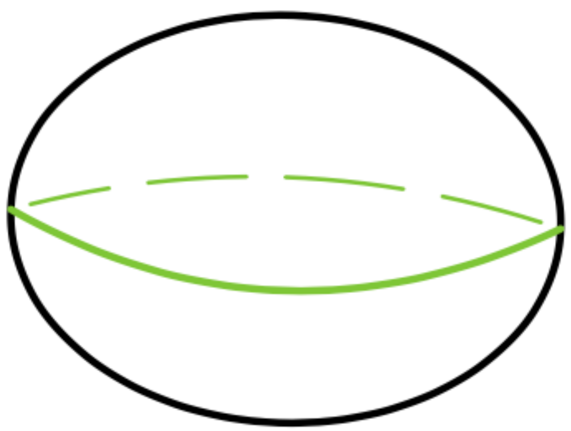

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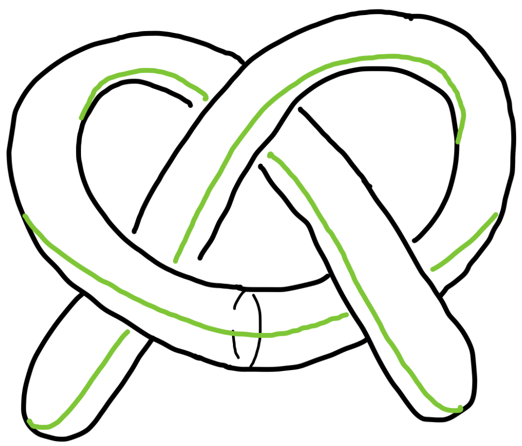

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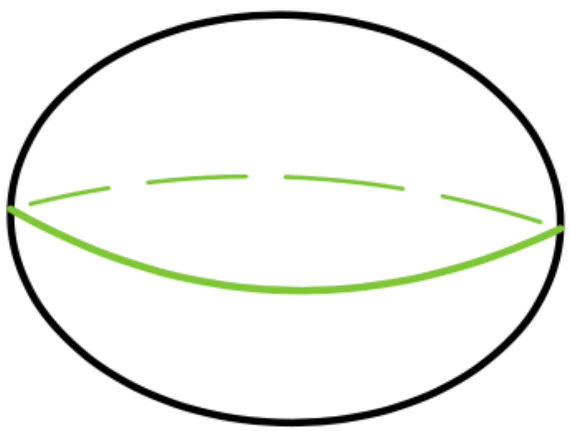

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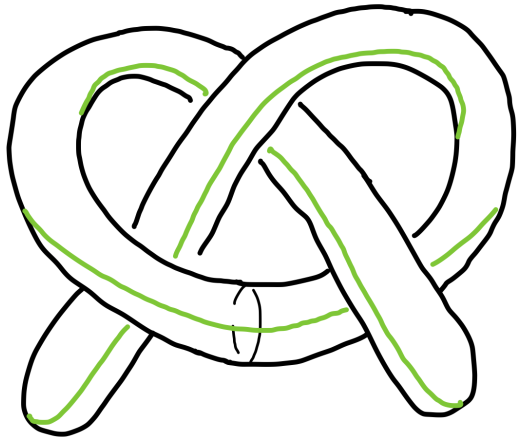

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A singular foliation $\mathscr{F}$ of S with leaves given by the level curves of a "horizontal direction" $h : \mathbb{R}^3 \to \mathbb{R}$.

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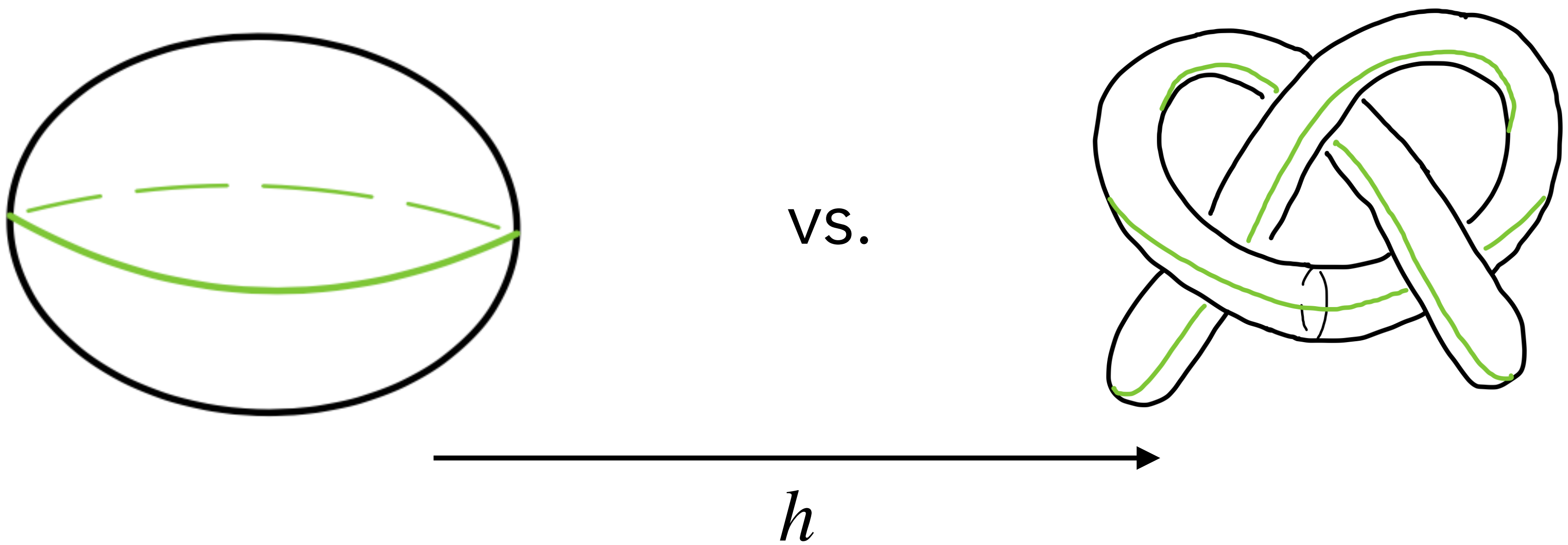

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$\mathscr{F}$ is transverse to $\mathscr{C}$ except at singularities

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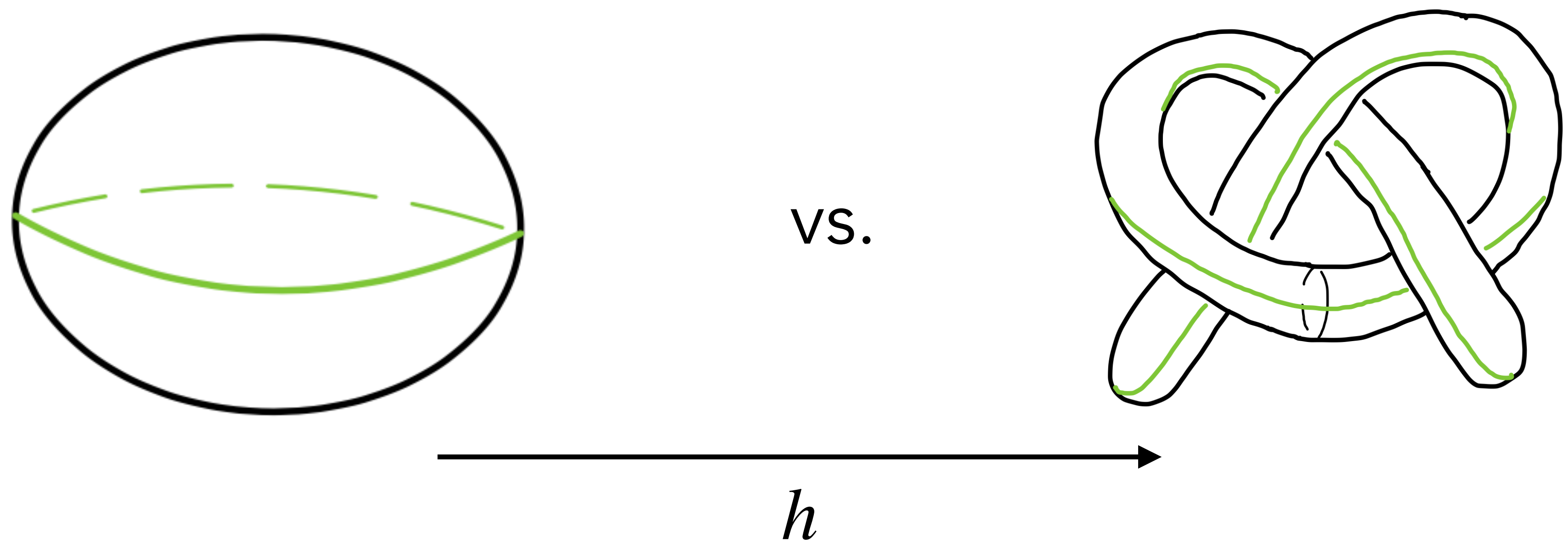
Count the singularities $s(\gamma)$ along $\gamma \in \mathscr{C}$:

$$s(\gamma) \geq 2\,|\,t(\gamma)\,|$$



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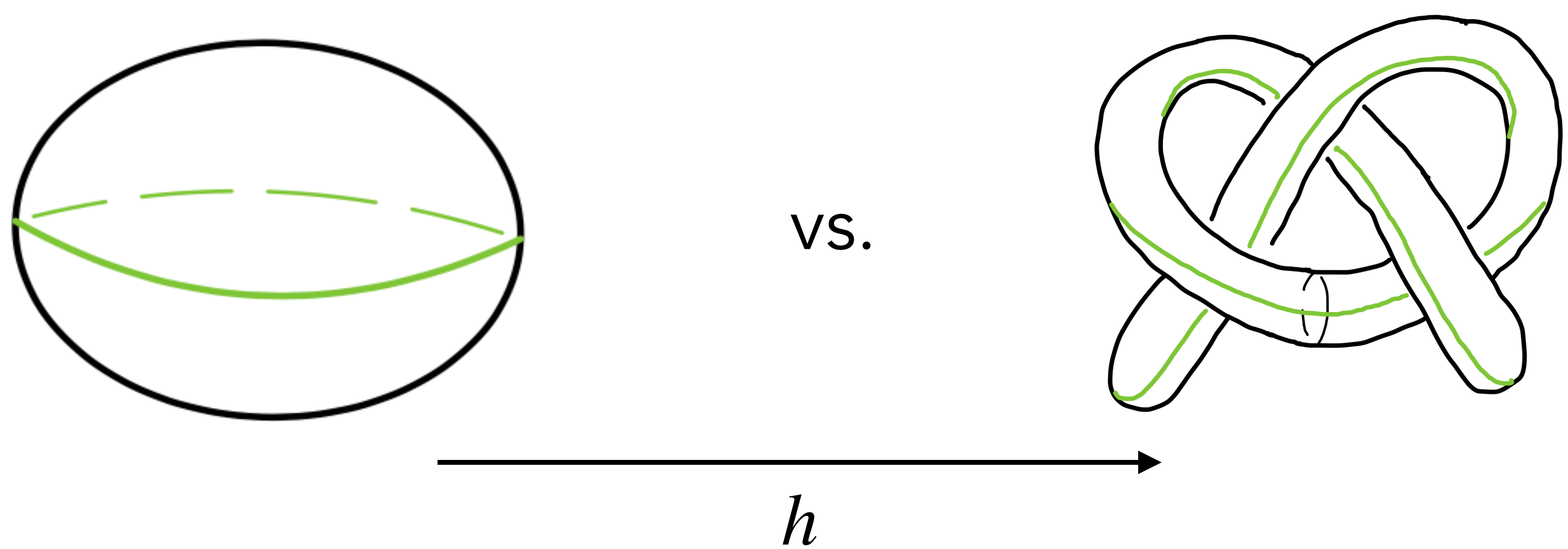

No excess crossings $\iff$ $s(\gamma) = 2\,|\,t(\gamma)\,|$ for all $\gamma \in \mathscr{C}$

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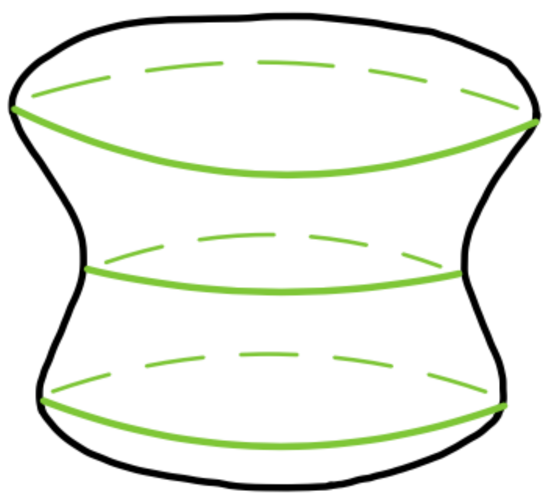
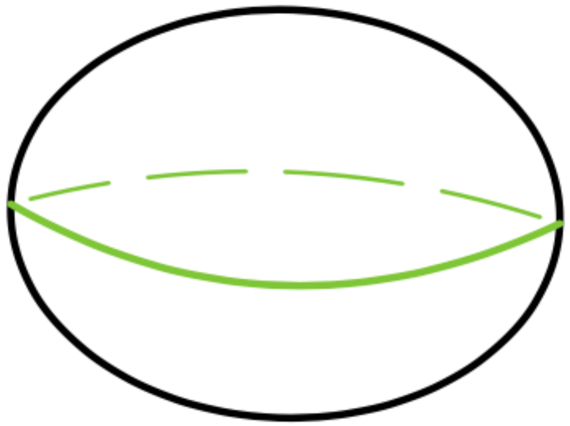
(With an eye toward interactions with an ambient link.)





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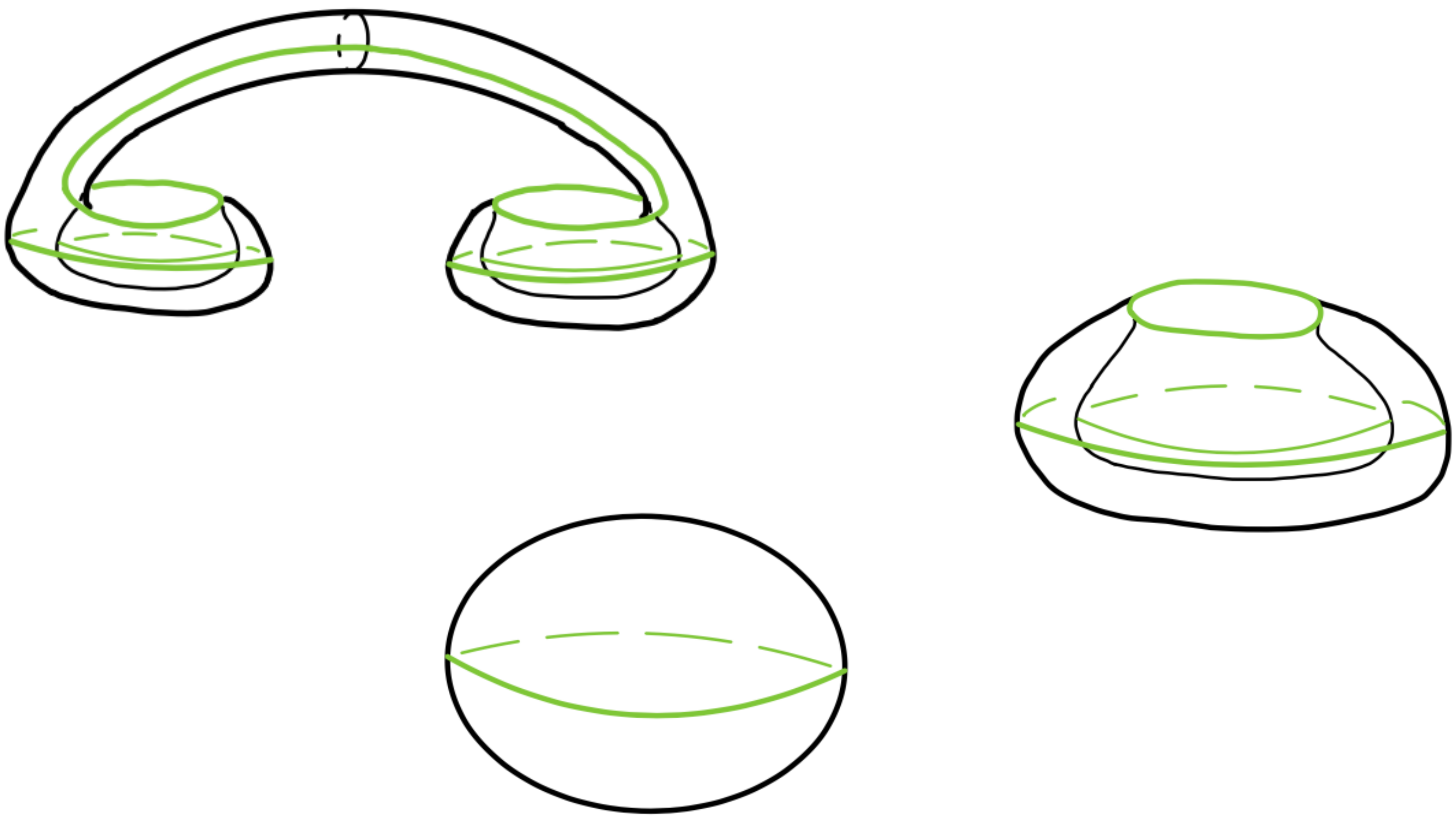

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